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Heisenberg Group as a Sub-Riemannian Manifold

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Heisenbergin ryhmä sub-Riemannin rakenteella varustettuna on malli geometrialle, joka esiintyy niin fysikaalisissa malleissa kuin eri puhtaan matematiikan osa-alueissa. Työssä tarkastellaan kolmiulotteista Heisenbergin ryhmää, ja keskitytään kuvaamaan sen geometriaa algebrallisten lausekkeiden avulla. Heisenbergin geometriaa voidaan pitää tasogeometrian yleistyksenä, minkä osoittaminen on eräs tämän työn tavoitteista.

Sub-Riemannin monistojen peruskäsitteistöä ja yleistä teoriaa esitellään lyhyesti. Tämän jälkeen määritellään sub-Riemannin rakenne Heisenbergin ryhmälle, ja esitetään avaruuden perusominaisuudet exponenttikoordinaattien avulla.

Heisenbergin ryhmä upotetaan kompleksiseen avaruuteen, jolloin Heisenbergin ryhmän algebrallinen sekä metrinen rakenne nähdään upotettuna tunnettuun avaruuteen. Samaistamalla Heisenbergin ryhmä erään kompleksisen hyperbolisen avaruuden hyperpinnan kanssa löydetään sub-Riemannin rakennetta säilyttävä kuvausluokka Heisenbergin ryhmälle. Konformikuvausten metristä teoriaa esitellään lyhyesti, ja todetaan tulos, joka osoittaa sub-Riemannin rakenteen nostavan tason Möbius-kuvaukset Heisenbergin ryhmän konformiryhmäksi.

Visuaalisen geometrian kielellä määritellään euklidisen eksponenttikuvauksen vastine Heisenbergin ryhmälle. Lopuksi kytketään hyperbolisen avaruuden visuaalisen geometria Heisenbergin ryhmän konformirakenteeseen.

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The Heisenberg group with its natural sub-Riemannian structure is a model for a geometry, which is encountered in applied models as well as areas of pure mathematics. In this thesis, we study the sub-Riemannian geometry of the three dimensional Heisenberg group through explicit computations. One of the main goals is to show how Heisenberg geometry can be viewed as a natural generalisation of plane geometry.

We briefly review the basic concepts and some general theory of sub-Riemannian manifolds. We then define the sub-Riemannian structure on the Heisenberg group, and outline its path metric properties using exponential coordinates.

The algebraic and the metric structure of the sub-Riemannian Heisenberg group can be seen embedded into complex hyperbolic space. Through identifying the Heisenberg group with an embedded hypersurface in the complex hyperbolic space, we obtain a class of conformal maps on the Heisenberg group. We briefly review some theory of conformal maps, and prove that the conformal maps on the sub-Riemannian Heisenberg group are lifted from Möbius maps on the plane.

We define a language of visual geometry and describe an analogue of the euclidean exponential map. Finally we find a connection between the visual geometry of the complex hyperbolic space and the Heisenberg group.

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Contents

1 Introduction

The topic of this thesis is the sub-Riemannian geometry of the first Heisenberg group. The approach that seemed to unify all facets of Heisenberg geometry was through complex hyperbolic geometry. While the focus will be on finding algebraic expressions for transformations and geometric objects on the Heisenberg group, some background theory of sub-Riemannian manifolds, complex geometry and quasiconformal mappings will be introduced to provide context. The Heisenberg group can be viewd as a natural noncommutative extension of plane geometry. A central theme will be the comparison of euclidean geometry and sub-Riemannian geometry of the Heisenberg group.

Sub-Riemannian geometry is most easily described in terms of control functions, and it arises in various applications. Section 2.1 introduces this viewpoint, and contains a brief summary of relevant metric geometry and the topology of sub-Riemannian manifolds. In section 2.3, the first Heisenberg group is introduced through its Lie algebra. We then define a canonical set of coordinates for the Heisenberg group in section 2.4.

Contact Heisenberg geometry is not particularly easy to 'see', unless one writes down algebraic formulae geometric objects, or parametrisations in explicit coordinates. This is also true for complex hyperbolic geometry, which is much more subtle than real hyperbolic geometry. Goldman's book [3] is an excellent reference for the many formulae it contains. Section 3.1 contains a concise introduction to complex hyperbolic geometry. Both the conformal and metric geometry of the sub-Riemannian Heisenberg group can readily be recovered from an embedding into the complex hyperbolic space, which is the subject of sections 3.2 and 3.3.

CR geometry is introduced the section 3.4, as it turns out to be the right algebraic language for describing the contact structure of the embedded Heisenberg group. In section 3.5 we provide an explicit one point compactification of the Heisenberg group as the boundary of the complex hyperbolic space. This yields us coordinate expressions for the natural transformations recovered from the embedding defined in section 3.2.

A classical application of the theory of conformal maps is the classification of conformal maps on $\mathbb{R}^n$ with $n > 2$, where the entire conformal group turns out to be generated by Möbius maps. We will make a slight digression in section 3.6 to give a short review of the metric theory of conformal maps on the Heisenberg group, based on a paper by Korányi and Reimann [1]. We find that the class of conformal maps is lifted from the Möbius group of the plane.

In section 4.1 we define a language of visual geometry for studying the metric and topological properties of the a space, with the point of view shifted from points to geodesics. This technique is then applied to the study of the metric structure of the sub-Riemannian Heisenberg group. Visual geometry of the complex hyperbolic space turns out to reveal a connection to the conformal geometry of the Heisenberg group. This connection is established in section 4.2.

1.1 A word on notation and language

Several approaches are used to describe the sub-Riemannian geometry of the Heisenberg group and consequently, while the language used in the thesis is mostly standard, concepts from several fields are used. The reader should have a knowledge of undergraduate differential geometry, algebra and analysis in several real and complex variables, including a familiarity with variational calculus. Essential terms, which are not standard vocabulary from these areas, will be defined within the text and highlighted in italics.

We will routinely identify two spaces by a bijective mapping F and use the same symbol for points and maps, which are F-related. This is commonplace in differential geometry, for instance, when one identifies points on a manifold and on euclidean space through coordinate charts.

Notation and basic theorems about differential geometry and Lie theory come from J. M. Lee's book [7]. Notation used in describing complex geometry is adopted from [3]. For instance, the euclidean inner product on $\mathbb{C}^2$ will be denoted by $(\cdot, \cdot)$, as the notation $\langle \cdot, \cdot \rangle$ is reserved for the indefinite inner product used in construction of the complex hyperbolic space.

Context specific omissions will be made when specifying objects and when listing the properties of explicitly defined objects. As we work predominantly in the smooth category, manifolds and all objects defined on them are smooth unless explicitly stated. This includes vector fields, forms, mappings and distributions. The terms curve and path are used synonymously to mean a piecewise smooth map from some sub-interval of $\mathbb{R}$ (one can usually assume it to be the unit interval $[0, 1]$) to a manifold M. The parameter of a path or its domain are usually not written out explicitly.

2 Sub-Riemannian Manifolds

2.1 The horizontal structure

Every statement in this section will be made for to three-dimensional manifolds, even though the general case is entirely analoguous. The aim is to establish the language of sub-Riemannian geometry before focusing on case of the Heisenberg group.

Let (M, H) be a pair consisting of a 3-manifold and a two-dimensional distribution called the *horizontal distribution* or *horizontal structure*. Define a local frame X, Y, Z such that the horizontal structure is spanned by X and Y, and Z gives the missing direction, or the *vertical direction*. A frame of this form is said to be *adapted* to the distribution. Lie-groups always admit left-invariant global frames, as left translation is a diffeomorphism. So for the rest of the section, we will assume that X, Y, Z is a global frame adapted to a horizontal structure H. We will keep the vector fields X and Y fixed, as the construction of sub-Riemannian manifolds will depend on this.

A curve $\gamma\colon [0,T] \longrightarrow M$ will be called *admissible*, or *horizontal*, if its derivative at t belongs to the space $H_{\gamma(t)}M$ for all $t \in [0,T]$. Equivalently we can require that the curve can be obtained as the integral curve of a *controlled* vector field of the form $u_1X + u_2Y$, where the *control functions* u_1 and u_2 are smooth. Note that L^1 integrability follows from smoothness in compact domains, so admissible curves are rectifiable in the sense that their sub-Riemannian length, which will be defined in the next section, is finite.

We will say points $p, q \in M$ are mutually accessible, if there exists an admissible path γ connecting p and q. We will denote this property by $\gamma\colon x \curvearrowright y$. The set of points accessible from point p will be called the accessible set and denoted by A_p. The smoothness requirement on admissible curves can be loosened into piecewise smoothness by a smoothing theorem that will be stated later. This is needed to allow the concatenation of curves needed in many constructions. An immediate consequence is that admissibility an equivalence relation.

Our first aim is to prove a remarkable theorem on accessibility called Chow's accessibility theorem, which gives us a local condition for the property of all points being pairwise mutually accessible. One can write this condition as

$$A_p = M$$

for all $p \in M$.

An obvious way to begin relating the concept of accessibility to the set of admissible points is to try to vary the control functions, which determine the path. This approach requires the use of an imbedding theorem on Banach spaces, which is beyond the level of this thesis. While proofs are omitted, the general ideas remain instructive. The discussion of the variation of functions is kept informal.

It is known from the theory of ordinary differential equations that the system

$$\begin{cases} \gamma' = u_1(t)X\gamma(t) + u_2(t)Y\gamma(t), \quad 0 \le t \le T \\ \gamma(0) = p, \end{cases} \tag{2.1}$$

has a unique solution. Also the system remains solvable, if the controls are changed slightly, in a way that that preserves their smoothness. Denote $u = (u_1, u_2)$ and denote the open neighbourhood of u in which system 2.1 is solvable by the symbol $U_{p,T} \subset C^\infty(\mathrm{Dom}(u_1, u_2), \mathbb{R})$. Note that we can always assume $T = 1$, as the scaling of u has the effect

$$\gamma_u(t) = \gamma_{Tu}(t/T)$$

on the path γ_u.

Define a map $E_p \colon U_{p,1} \longrightarrow M$ by setting $E_p(u) = \gamma_u(1)$, where γ_u is the solution to the system 2.1. This map is called the end-point map. By showing that the map E_p is of constant rank, one is led to the following result known as the Orbit theorem.

Proposition 2.1. *The accessible set A_p of a associated to a distribution is an immersed submanifold for any $p \in M$. [2].*

This yields Chow's accessibility theorem as an easy corollary. The distribution H is said to be *bracket generating*, if the fields X, Y and $[X, Y]$ span the tangent space at every point. A bracket generating distribution is maximally nonintegrable in the sense that the vector $[X, Y]$ is not in H at any point.

Corollary 2.1. *Let M be a connected manifold with a horizontal subbundle H. Assume that the distribution H is bracket generating. Then all points of M are pairwise accessible.*

Proof. Choose a point $p \in M$ and a local frame X, Y, Z adapted to the horizontal structure in the neighbourhood of p. As A_p is an immersed submanifold, the bracket $[X, Y]_q$ is in the tangent space T_qA_p for every point $q \in A_p$. So because H is bracket generating, we must have $T_qA_p = T_qM_q$. Hence the subset A_p is open in M. Because M is connected and accessibility components form a partition of M, we must have $A_p = M$. $\square$

Next, we will prove the same result by using properties of flows. Recall that the flow ϕ of a vector field V is a smooth mapping $M \times D \longrightarrow M$, where D is an interval containing 0, that satisfies

$$\frac{d\phi}{dt} = V,$$

and

$$\phi(p, 0) = p$$

for all $p \in M$. It is easy to show that in local coordinates the flow ϕ has the form

$$\phi = id + tV + \mathcal{O}(t^2). \tag{2.2}$$

The commutator of two flows ϕ and θ of vector fields V and W is defined as

$$[\phi(t), \theta(t)] = \phi(t)\theta(t)\phi(-t)\theta(-t)$$

By substituting equation 2.2 we one readily finds that the first order approximation of the map defined in equation 2.1 is the identity. By computing second derivatives of the expression, one finds the Taylor expansion

$$[\phi(t), \theta(t)] = id + t^2[V, W] + \mathcal{O}(t^3)$$

for the commutator of flows in coordinates adapted to the frame V, W.

Proposition 2.2. *Let M be a connected sub-Riemannian 3-manifold. Then all points of M are pairwise accessible.*

The following argument can be generalised to higher dimensional cases by induction.

Proof. Fix a point $p \in M$. It suffices to show that the set A_p is open to show that $A_p = M$, since M is connected.

Let X, Y be a frame adapted to H in some neighbourhood U of p. Identify U with a neightbourhood of 0 such that p is identified with 0.

Denote the flows of X and Y by ϕ and θ, respectively. Denote the bracket of X and Y by Z. By assumption these form a local trivialisation of TU. In the coordinates adapted to this frame we have

$$\phi(t) = \mathrm{id} + tX + \mathcal{O}(t^2), \quad \theta(t) = \mathrm{id} + tY + \mathcal{O}(t^2).$$

Set

$$\bar{\phi}(t) = \begin{cases} \phi(\sqrt{t}) & \text{if } t \geq 0, \\ \phi(-\sqrt{t}) & \text{if } t < 0, \end{cases} \quad \bar{\theta} = \begin{cases} \theta(\sqrt{t}) & \text{if } t \geq 0, \\ \theta(\sqrt{t}) & \text{if } t < 0. \end{cases}$$

The commutator of the flows defined above satisfies

$$[\bar{\phi}(t), \bar{\theta}(t)] = \mathrm{id} + t[X, Y] + \mathcal{O}(t^2).$$

The composition of the above with the flows ϕ and θ yields a map $U \times A \longrightarrow M$, where A is a subset of $\mathbb{R}^3$ containing the origin, defined by the formula

$$\begin{aligned}\varphi(t_1, t_2, t_2) =& \phi_t \circ \theta_{t_2} \circ [\bar{\phi}_{t_3}, \bar{\theta}_{t_3}] \\ =& \mathrm{id} + t_1 X + t_2 Y + t_3[X, Y] + \mathcal{O}(t^2).\end{aligned}$$

The differential of this map is clearly invertible in some neighbourhood of 0, so we find an open neighbourhood V of $\mathbb{R}^3$ in which φ is a diffeomorphism. The image $\varphi(V)$ is accessible from p, since the p moves along a concatenation of horizontal curves. We have shown that $TA_p = TM$, which proves that p is an interior point of A_p. □

When all points are mutually accessible, the horizontal structure is called a contact structure.

The contact structures can conveniently be given as the kernel a 1-form. A *calibrating form* ω for a distribution H is a 1-form which satisfies $\mathrm{Ker}\omega = H$. A calibrating form ω is a *contact form*, the wedge product

$$\omega \wedge d\omega$$

is everywhere non-zero. Accessibility has a simple characterisation in terms of contact forms.

Proposition 2.3. *Let η be a 1-form. Then the distribution defined as the kernel of η is integrable if and only if the form η satisfies the equation*

$$\eta \wedge d\eta = 0.$$

At the other extreme, if η is a contact form on M, then its kernel satisfies Chow's condition.

Proof. Suppose η defines the horizontal distribution H on a 3-manifold M and let X, Y be horizontal vector fields. Integrability can be characterised by Frobenius' theorem [7, Theorem 14.5] as the equality

$$0 = \eta(X) = \eta(Y) = \eta([X, Y]).$$

We can see from the identity

$$d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y])$$

that H is integrable if and only if $d\eta(X,Y) = 0$. We have the equality

$$\eta \wedge d\eta(X,Y,Z) = C\left(\eta(X)d\eta(Y,Z) + \eta(Y)d\eta(Z,X) + \eta(Z)d\eta(X,Y)\right)$$

for some positive constant C. If H is integrable, the above is identically zero. If, on the other hand, the last term is not zero, we must have $\eta([X,Y]) \neq 0$, which means that $X, Y, [X,Y]$ spans the tangent bundle. One only needs to choose X and Y so that they span the horizontal distribution. $\square$

Finally we state the smoothing theorem mentioned earlier.

Proposition 2.4. *[4, 1.2.B] Let (M,H) be a contact 3-manifold and $\gamma\colon p \curvearrowright q$ is a piecewise smooth curve connecting p and q, which are points on M. There exists a smooth curve connecting p and q.*

2.2 The Carnot-Carathéodory metric

Assume the vector fields X and Y define a two-dimensional distribution on a manifold M, and that the Lie bracket $Z = [X,Y]$ spans the missing direction. The global frame X, Y, Z is held fixed for the rest of this section.

Let p and q be points on M and denote by γ_u the integral curve given by controls u_1 and u_2. Define the Carnot-Carathéodory (CC) length $L(\gamma_u)$ of γ_u by the integral

$$L(\gamma_u) = \int \left(u_1^2 + u_2^2\right)^{\frac{1}{2}}, \tag{2.3}$$

over the parameter interval $\mathrm{Dom}\gamma$. The *length structure* determined by the function L satisfies a natural collection of axioms, including the following.

- Path length is additive with respect to concatenation of horizontal curves.
- Path length is increasing, continuous and smooth with respect to the parameter.
- Path length is invariant in smooth reparametrisations.

The CC distance $d_C(p,q)$ of points $p, q \in M$ is computed from the length structure as the lower bound $\inf_u L(\gamma_u)$, where the path γ_u is controlled by u and connects p and q.

We gave the construction in terms of the special frame X, Y, Z, but the CC metric can equivalently be constructed by giving a positive definite quadratic form Q on the horizontal bundle, which assigns the value ∞ to all nonhorizontal vectors. The CC length of a horizontal path γ is defined by

$$L(\gamma) = \int \sqrt{Q(\gamma')}.$$

The definition can be extended to all paths on M by setting $Q(V) = \infty$ for vectors with a vertical component. The quadratic form can be obtained from the earlier definition by setting Q to be the unique quadratic form that makes X, Y an orthonormal system and assigns the value ∞ to nonhorizontal vectors. The distance function is again defined as the lower bound for the length of connecting paths. In coordinates adapted to the frame X, Y, Z, natural guess for a matrix representation, which approximates the quadratic form, would be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & L \end{pmatrix},$$

where L is taken to infinity. We will return to an approximation of this kind in section 3.3, but the coordinate frame will be related to the Lie group structure.

Recall that a *geodesic* (arc or segment) between points p and q on a Riemannian 3-manifold is a nonconstant solution to the geodesic equation

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \tag{2.4}$$

given in terms of local coordinates x^i, the Christoffel symbols Γ^i_{jk} of the Riemannian metric and the boundary conditions determined by points p and q. Equivalently, geodesics are the critical points of the first variation on length by curves connecting p and q, that is, they are locally length minimising. *Geodesic completeness* is the property that all geodesics can be infinitely extended. On Riemannian manifolds we have the following classical theorem.

Proposition 2.5 (Hopf-Rinow)**.** *[13] Let V be a Riemannian manifold. The following are equivalent.*

- *The manifold V is (metrically) complete.*
- *The manifold V is geodesically complete.*
- *Any two points on V can be connected by a length minimizing geodesic.* □

On a sub-Riemannian manifold, the term geodesic will be used to mean a length minimising curve connecting two points p and q parametrised by a nonzero multiple of the CC arc length (usually unit speed). Geodesics between two points p and q on sub-Riemannian manifolds can also be defined by a geodesic equation similar to the Riemannian case, and as critical points of length for horizontal variations. Details for this formulation of geodesity on general sub-Riemannian manifolds can be found in the paper [14] and explicit calculations on the Heisenberg group are done in the book [12].

A statement analoguous to Proposition 2.5 is true for sub-Riemannian manifolds.

Proposition 2.6. *[14] Suppose M is a sub-Riemannian manifold.*

- *If M is complete, then every geodesic can be extended infinitely, and any two points can be joined by a geodesic.*
- *If for some point $p \in M$, all geodesics starting from p can be infinitely extended, then M is complete.*

□

In later sections, we will study the topological properties mentioned above on the Heisenberg group.

2.3 The Heisenberg Lie algebra

This section introduces the first Heisenberg group, which is the simplest nontrivial example of a sub-Riemannian manifold. We will define a contact structure based on the noncommutativity of the group structure. This contact structure is often called the standard contact structure of $\mathbb{R}^3$.

Start from a three dimensional vector space with a basis V_1, V_2, V_3 and define the bracket relations

$$[V_1, V_2] = -4V_3, \; [V_1, V_3] = [V_2, V_3] = 0.$$

We will denote this Lie algebra by $\mathfrak{h}$. Note that $\mathfrak{h}$ is step two-nilpotent, which means that every expression containing a nested bracket evaluates to zero.

The correspondence theorem between Lie algebras and simply connected Lie groups [7, Theorem 15.35] implies that there is a unique group structure in $\mathbb{R}^3$, whose Lie algebra is $\mathfrak{h}$. We will call this group the first Heisenberg group. The contact structure on the first Heisenberg group can now be described in terms of its Lie algebra. The vectors V_1 and V_2, interpreted as vectors in $T_0\mathbb{H}$, span the plane $H_0\mathbb{H}$. Left-translation allows us to define a two-dimensional distribution H on $\mathbb{H}$.

The commutator of V_1 and V_2 spans the missing direction at the origin, so by left invariance the distribution is bracket generating. The sub-Riemannian distance function d_C on the Heisenberg group can now defined by the construction given in the previous section, using the invariant vector fields V_1 and V_2 as the control fields. The term Heisenberg group and the symbol $\mathbb{H}$ will, from now on, refer to the sub-Riemannian manifold $(\mathbb{H}, H)$ with the intrinsic distance function d_C. This distance function was defined in terms of the invariant vector fields, so left translation will define a d_C isometry. Hence the isometry group will be transitive.

Coordinates will be given in section 2.4. The sub-Riemannian Heisenberg group models the geometry of the plane at each point. We will study this connection to planar geometry in the following sections. The Lie algebra of the euclidean plane can be seen as the image of the projection onto $V_1 \oplus V_2$. The metric structure will reveal connections to both real euclidean and complex hyperbolic spaces.

2.4 Exponential coordinates on the Heisenberg group

To define coordinates on $\mathbb{H}$, we will identify points in the Heisenberg group with points in its Lie algebra. This is possible because of the following proposition.

Proposition 2.7. *[5] Let G be a simply connected analytic Lie group whose Lie algebra is nilpotent. Then the exponential map is a diffeomorphism.* □

The exponential map $\mathfrak{h} \longrightarrow \mathfrak{H}$ defined on the Heisenberg Lie algebra will be denoted by $X \mapsto e^X$ and the left action of e^X on a point p by $e^X p$.

A point $W \in \mathfrak{h}$ is of the form $W = (xV_1 + yV_2 + vV_3)$, which we will identify with a point $(x, y, v) \in \mathbb{R}^3$. We continue to identify the point $(x, y, v) \in \mathbb{R}^3$ with its image under the exponential map, so we get the identification given by

$$(x, y, z) \longleftrightarrow e^{xV_1 + yV_2 + vV_3}.$$

These coordinates will be referred to as *exponential coordinates* on the Heisenberg group. We will frequently use complex notation and write $(z, v) = (x + \mathrm{i}y, v) \in \mathbb{C} \times \mathbb{R}$ for (x, y, v).

To derive the group law for $\mathbb{H}$ in exponential coordinates, we begin by writing the Baker-Campbell-Hausdorff formula [5] in the step two-nilpotent case

$$pq = \log\left(e^W e^{W'}\right) = W + W' + \frac{1}{2}[W, W'].$$

Substituting W and W' in the V_1, V_2, V_3-basis, the above expression takes the form

$$(x + x')V_1 + (y + y')V_2 + (v + v')V_3 + \frac{1}{2}(xV_1 + yV_2 + vV_3, x'V_1 + y'V_2 + v'V_3).$$

The commutator relation given by equation 2.3 of $\mathfrak{h}$ yields the group law for points $W = (z, t), W' = (w, s) \in \mathbb{H}$

$$(z, t)(w, s) = (z + w, t + s + 2\,\mathrm{Im}\, z\bar{w}). \tag{2.5}$$

In terms of real coordinates, the group law for elements (x, y, v) and (x', y', v') can be expressed as

$$(x, y, v)(x', y', v') = (x + x', y + y', t + s + 2(x'y - xy')..$$

Let $p \in \mathbb{H}$ be a point with coordinates (x, y, v). The derivative map of the left translation $L_p(q) = pq$ can now be deduced from equation 2.4.

$$(L_p)_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2y & -2x & 1 \end{pmatrix}$$

Now consider the coordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial v}$. Translation invariant vector fields can be obtained from these by applying the linear map $(L_p)_*$ to produce a global frame given by

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial v}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial v} \quad \text{and} \quad \nu = \frac{\partial}{\partial v}. \tag{2.6}$$

The two dimensional distribution X, Y spans the horisontal distribution $H\mathbb{H}$ or H for short. This distribution is given by the 1-form

$$\omega = dv + 2\,(xdy - ydx) = dv + 2\,\mathrm{Im}\, zd\bar{z}.$$

A direct computation shows that ω is a contact form:

$$\begin{aligned} \omega \wedge d\omega &= (dv + 2\,(xdy - ydx))) \wedge (2dx \wedge dy - 2dy \wedge dx) \\ &= -\,4dv \wedge dy \wedge dx = 4dx \wedge dy \wedge dv, \end{aligned}$$

so $\omega \wedge d\omega$ is nondegenerate.

The vector fields X and Y are identified with the vectors V_1 and V_2 in the Lie algebra. Hence the sub-Riemannian length structure is now determined by these two vector fields.

A horizontal path γ on $\mathbb{H}$ given by controls $a, b \in C^\infty(I, \mathbb{R}^2)$ can be used to define a path on the plane satisfying $(x', y') = (a, b)$ called the *projection* of γ. This path has the important property, that the euclidean length of the projected path is the same as the CC length of the original path. Conversely, a path $\alpha = (a, b)$ can be lifted uniquely to a path $\hat{\alpha}$ the Heisenberg group, as the third component is determined by the horizontality condition $\hat{\alpha}' \in \mathrm{Ker}(\omega)$ on the tangent of $\hat{\alpha}$. The expression for the unique lift is

$$\hat{\alpha} = \left(a(t), b(t), -2 \int_I (ab' - ba')(t) dt \right).$$

A rather remarkable property of the exponential coordinates is that we can easily derive simple parametrisation for geodesic arcs. Recall the following classical theorem.

Proposition 2.8. *[8] Suppose $\Phi_A(p)$ is the family of closed plane curves, with a common point p that enclose the area A. The circle of radius A/π is a critical point of length, and the unique length minimising curve.*

Corollary 2.2. *Suppose $\Phi_A(p,q)$ is the family of injective plane curves obtained by concatenating a smooth curve with a segment $[q,p]$ such that their enclosed areas are equal to A. The circular arc is the unique critical point of length, and the unique curve of minimum length. When $A = 0$ the only critical point is a straight line.*

Proposition 2.9. *CC geodesics in $\mathbb{H}$ are lifts of constant curvature plane curves.*

Left translation preserves the geodesity of curves, so we may assume the initial point of a geodesic arc to be the origin.

Proof. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)\colon [0,p] \longrightarrow \mathbb{H}$ be a geodesic arc in $\mathbb{H}$ connecting points 0 and $p = (x, y, t)$. Consider the concatenation of the curve (γ_1, γ_2) with the straight line segment connecting its end points. Call this piecewise smooth curve α. Suppose first that α has no self-intersections. We write t in terms of derivatives and use the horizontality of γ

$$\begin{aligned} t &= \int_0^1 \gamma_3'(t) dt \\ &= -2 \int_0^1 (\gamma_1\gamma_2' - \gamma_2\gamma_1') dt \end{aligned}$$

The integrand above $xdy - ydx$ will equal to zero along any straight line segments. Therefore the integral is unchanged in the concatenation and we have

$$t = -2 \int_\alpha xdy - ydx.$$

The path α is a piecewise smooth curve bounding a surface S in the plane, so we can use Stokes' theorem to find

$$t = -4\int_\alpha dx \wedge dy \tag{2.7}$$

$$= -4\text{Area}(S), \tag{2.8}$$

Proposition 2.8 requires the projection to be a curve of constant curvature, which satisfies the area and end point requirements. We also see that if (γ_1, γ_2) is a straight line segment, it is a geodesic, by Corollary 2.2.

Now suppose the geodesic γ is arbitrary and the curve α is allowed self-intersections. Choose $s \in [0,1]$ and suppose the curvature of of projection $\beta = (\gamma_1, \gamma_2)$ is positive at this point. Curvature is a continuous function by the smoothness of γ, so we can choose a neighbourhood $[a,b]$ where the curvature does not change its sign. The restriction of β to the subinterval $[a,b]$ is geodesic and by the above argument, of constant curvature. Suppose, on the other hand that the curvature is zero at some point. The curvature of β is continuous by the smoothness of γ, so if the curvature is zero at any time $s \in I$, then it must be zero in some neighbourhood of s. Hence the curvature of β is locally constant, which by continuity of β implies that it is indeed constant. Hence γ is the lift of a constant curvature path in $\mathbb{R}^2$. □

The solution to the isoperimetric problem in the plane also gives us the following.

Corollary 2.3. *Length realising curves between points (z,v) and $(0,0)$ in $\mathbb{H}$ are lifts of unique circular arcs if $z \neq 0$, and otherwise they are lifted from the family of circles of fixed radius containing the origin.*

Proof. The projection of a length realising curve must either be in the family defined in Proposition 2.8 or after concatenation by a segment in the family defined in Corollary 2.2. This implies the result. □

The family of lifts starting from the origin can be parametrised in terms of the radius R of the projection, the angle parameter t corresponding to the vector $\cos(t)\frac{\partial}{\partial x} + \sin(t)\frac{\partial}{\partial y}$ and the length parameter $s \in \mathbb{R}^+$ as

$$(t,R,L) \longleftrightarrow \left(R(e^{\mathrm{i}(s-\pi/2)} \pm \mathrm{i}e^{\mathrm{i}t}), \pm 2R^2\left(\sin(s) - s\right)\right)_{s\in[0,L/R]}$$

and

$$(t,R,L) \longleftrightarrow \left(se^{\mathrm{i}t}, 0\right)_{s\in[0,L]}.$$

The uniform limit of this family when R tends to infinity is the straight line with the initial tangent vector $\cos(t)\frac{\partial}{\partial y} - \sin(t)\frac{\partial}{\partial x}$. It is often more convenient to

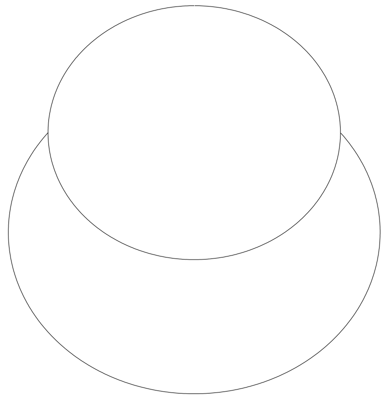

FIGURE 1: *Two curves* $(0,0) \curvearrowright (1,0)$ *that lift to geodesics of distinct length on* $\mathbb{H}$.

express the family given by 2.4 in terms of the curvature $c = R^{-1}$ and the unit speed parametrisation obtained by replacing s with cs.

$$(t, c, L) \longleftrightarrow \left(\frac{1}{c} (\left(e^{\mathrm{c}\mathrm{i}(s-\pi/2)} \pm \mathrm{i}e^{\mathrm{i}t} \right), \pm \frac{2}{c^2} \left(\sin(cs) - cs \right) \right)_{s \in [0,L]}.$$

This is the parametrisation of geodesic arcs we will use in the following.

Example 2.1. Consider geodesics connecting the origin to the point $(1, 0, -6\pi)$. One such example is the lift of the circle of radius 0.5 with the center $(\frac{1}{2}, 0)$ transversed one and a half times in the positive direction. We know that there is a shorter path, which can be lifted from the circular arc of radius R and complement angle 2α. From elementary geometry, we have the pair of equations

$$2R \sin \beta = 1,$$
$$\pi R^2 (1 - \frac{\beta}{2\pi}) + R \cos \beta = 3/2\pi,$$

whose solution is $R = 0.6357, \beta = 0.9052$. The center is at the point $(0.5, 0.3926)$. Denote the two geodesics aquired by lifting the larger and the smaller circular arc by γ_1 and γ_2. The existence of a third geodesic would require the projected circle to be traversed more than twice. Hence these are the only geodesics, as γ_2 is obtained by lifting the smallest circle containing the points $(0, 0)$ and $(1, 0)$. The CC lengths of the two geodesics are related by

$$L(\gamma_2) = 1.6573 L(\gamma_1).$$

and the ratio of their curvatures is 1.2714.

Let q be the point $(z, s) \in \mathbb{H}$ and call the quantities $|z|$ and $|s|$, respectively, the horizontal and vertical distance between 0 and q. For general pairs of points $p, q \in \mathbb{H}$, define the vertical (horizontal) distance between them to be the vertical (horizontal) distance between $p^{-1}q$ and 0. We will give names to the two extremal relative positions of points in $\mathbb{H}$. If the vertical (horizontal) distance from p to q is zero, the points are said to be in horizontal (vertical) relative position. In euclidean space, one often makes the identification between a tangent plane and an affine subspace. We can do similar identifications on the Heisenberg group. For instance, the horizontal plane at point p spans an affine subspace of $\mathbb{H}$ of points, which are accessible by zero-curvature geodesics.

The added third dimension perpendicular to the horizontal plane has peculiar properties, which now become quite tangible, as the simple parametrisations turn the study of geodesics into simple analytic geometry. We are ready to prove the following proposition.

Proposition 2.10. *Suppose t is nonzero. If the ratio $|t|/|z|^2$ is finite, there are a finite number of geodesics connecting* 0 *and* (z, t).

Proof. The minimum radius of a connecting geodesic is given by $|z|/2$, as seen from the above parametrisations. Hence, by equation 2.7, we have the inequality

$$k \leq \frac{v}{\pi|z|^2}$$

on the number of geodesics k. $\square$

Example 2.2. As an example of the vertical extremal case, we provide an infinite family of geodesics of distinct length connecting the points $(0, 0)$ and $(0, 1)$ in $\mathbb{H}$ in complex form. Let γ_k denote the lift of the path

$$\frac{1}{2\sqrt{k\pi}}(e^{-i\phi} - 1).$$

To see that γ_k connects points $(0, 0)$ and $(0, 1)$, notice that the enclosed area of each curve is 1. The CC length of curve γ_k is $\sqrt{\pi/k}$.

It is clear that geodesics of the form 2.4 and 2.4 can be infinitely extended, so we have metric completeness by Proposition 2.6. Another way to see that $\mathbb{H}$ is metrically complete is introduced in the next section. Section 3.3 will provide yet another technique for proving completeness, as well as other topological properties, by Riemannian approximation.

2.5 The Heisenberg metric

This section introduces an easily computed metric on the Heisenberg group that is bilipschitz-equivalent to the sub-Riemannian metric defined earlier. Define the Korányi gauge $|p|_{\mathbb{H}}$ for $p \in \mathbb{H}$ by

$$|(z,t)|_{\mathbb{H}} = \left(|z|^4 + t^2\right)^{\frac{1}{4}}.$$

Now we can define the distance between (z,t) and (w,s) by

$$d_{\mathbb{H}}\left((z,t),(w,s)\right) = |(z,t)^{-1}(w,s)|_{\mathbb{H}} = |(w-z, s-t-2\operatorname{Im} z\bar{w})|_{\mathbb{H}}.$$

Proposition 2.11. *The function $d_{\mathbb{H}} \colon \mathbb{H} \longrightarrow \mathbb{R}$ is a metric.*

Proof. Clearly $d_{\mathbb{H}}$ is zero if and only if (z,t) and (w,s) are equal. It is symmetric, since $|(z,t)^{-1}|_{\mathbb{H}} = |-z|^4 + (-t)^2 = |(z,t)|_{\mathbb{H}}$. The inequality

$$d_{\mathbb{H}}(p,q) \leq |p|_{\mathbb{H}} + |q|_{\mathbb{H}}$$

was proven in [1, sec F]. Hence for any triplet $p,q,r \in \mathbb{H}$, we have

$$\begin{aligned} d_{\mathbb{H}}(p,r) + d_{\mathbb{H}}(q,r) =& |p^{-1}r|_{\mathbb{H}} + |q^{-1}r|_{\mathbb{H}} \\ \geq& |(p^{-1}r)^{-1}q^{-1}r|_{\mathbb{H}} \\ =& |(r^{-1}p)(q^{-1}r)|_{\mathbb{H}} \\ =& |(q^{-1}r)(r^{-1}p|_{\mathbb{H}} \\ =& d_{\mathbb{H}}(p,q), \end{aligned}$$

which proves the triangle inequality. Other metric axioms are clear from the definition. □

The metric $d_{\mathbb{H}}$ will be called the *Heisenberg metric*, while we use the term Heisenberg CC metric for the distance function d_C.

Path length of continuous paths can be computed without reference to derivatives if the distance function is given.

Definition 2.1. The path length of a continuous path γ with regard to a given metric d is given by

$$L(\gamma) = \limsup_{N\to\infty} \sum_{i=0}^{N} d(\gamma(t_{i+1}), \gamma(t_i)),$$

where the supremum is taken over all finite subsets $\operatorname{Dom}(\gamma)$ such that $t_i < t_{i+1}$.

By this definition, the metric $d_{\mathbb{H}}$ induces the same length structure as the metric d_C.

Proposition 2.12. *[8] Let $\gamma\colon [0,1] \longrightarrow \mathbb{H}$ be a path. The length structures induced by d_C satisfies*

$$L_{d_{\mathbb{H}}}(\gamma) = \begin{cases} L_C(\gamma) & \text{if } \gamma \text{ is horizontal,} \\ \infty & \text{otherwise.} \end{cases}$$

□

The Heisenberg metric is an example of a non intrinsic metric. The induced length metric given by the length structure $L_{d_{\mathbb{H}}}$ is not the same metric as the metric $d_{\mathbb{H}}$ itself. In other words, it $d_{\mathbb{H}}$ is not the same function as its induced length metric d_C. The metric space $(\mathbb{R}^3, d_{\mathbb{H}})$ is clearly complete and locally compact, so the infimum can be taken to be a minimum that is achieved by some path γ. Pairs of points in $(\mathbb{H}, d_{\mathbb{H}})$ that are not in horizontal position relative to each other have no distance realising geodesics connecting them. In other words, no curve that satisfies the equation $d_C(p,q) = L_{d_{\mathbb{H}}}(\gamma)$ exists. Still, the metrics are equivalent and we have the inequality

$$\frac{1}{\sqrt{\pi}} d_C(x,0) \leq d_{\mathbb{H}}(x,0) \leq d_C(x,0).$$

The inequalities are tight. To see the tightness of the lower bound, consider the points $p = (0,0)$ and $q = (0,1)$ in $\mathbb{H}$. The length realising curve is the lift of a circle of radius $\frac{1}{2\sqrt{\pi}}$, hence the CC distance is equal to $\sqrt{\pi}$. So the ratio of metrics satisfies

$$\frac{d_{\mathbb{H}}(p,q)}{d_C(p,q)} = \frac{1}{\sqrt{\pi}}.$$

For the upper bound, consider points in relative horizontal position, where both metrics restrict to the euclidean metric.

The upper bound follows from Proposition 2.12, since the induced length metric always majorises the original metric. The left inequality is easy to justify heuristically by computing the ratio of distances for pairs of points with different vertical distance to squared horizontal distance ratios. However, a proof for the tight lower bound could not be found. The CC metric functions involves transcendental functions, and it seems tricky to try to find the right inequalities, which would lead to a tight bound. We will sketch a proof without the tight bound in section 3.5. The maximum is reached when the two points are in vertical relative position.

The metric completeness of $(\mathbb{H}, d_{\mathbb{H}})$ is clear, so $(\mathbb{H}, d_C)$ is complete by the equivalence of metrics.

3 Complex Hyperbolic Space and the Heisenberg Group

3.1 The ball model

In this section we will introduce the complex hyperbolic space and outline some of its basic properties. The complex hyperbolic space will serve as an ambient space for the Heisenberg group. Recall that the construction of the real hyperbolic 3-space leads to the unique homogeneous, isotropic manifold of constant sectional curvature of the specified dimension. The construction of the complex hyperbolic space is the same, but the metric structure of the complex hyperbolic space is more involved. The Möbius maps of the plane are obtained from the action of the real hyperbolic group (the isometry group of the real hyperbolic plane). A similar connection can be found between the Heisenberg group and a subgroup of the complex hyperbolic group. An outline of the basic properties of real hyperbolic space can be found in [15]. Unlike in the case of the construction of real hyperbolic space, the sectional curvature is non-constant. With our choice of scale, the sectional curvature of $\mathbb{C}\mathbf{H}^2$ ranges between -1 and $-\frac{1}{4}$ [3].

We start by defining the space of complex lines $\mathbb{P}(\mathbb{C}^3)$ consisting of points $[v] \in \mathbb{P}(\mathbb{C}^3)$ that are one-dimensional subspaces spanned by nonzero vectors $z \in \mathbb{C}^3$. This space is called the complex projective space. The space $\mathbb{P}(\mathbb{C}^3)$ is a complex manifold with smooth structure given by the holomorphic quotient map $\mathbb{P}\colon z \mapsto [z]$, with $[z]$ denoting the complex line represented by a vector $z \neq 0$. The map $\mathbb{P}$ is often called the projectivisation. We will use the shorthand $\mathbb{P}^2$ for $\mathbb{P}(\mathbb{C}^3)$.

Let $\langle \cdot, \cdot \rangle$ denote the quadratic form defined by the matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

so for vectors z, w in $\mathbb{C}^3$, it has the expression

$$\langle z, w \rangle = \bar{z} J w = z_1 \bar{w}_1 + z_2 \bar{w}_2 - z_3 \bar{w}_3.$$

We will call vectors $z \in C^3$ negative, if $\langle z, z \rangle < 0$ and define null and positive vectors similarly by conditions $\langle z, z \rangle = 0$ and $\langle z, z \rangle > 0$, respectively. The space $\mathbb{C}^3$ endowed with this bilinear form is often denoted by $\mathbb{C}^{2,1}$. Negativity is preserved under dilations $z \mapsto \lambda z$, so it is well defined for elements of $\mathbb{P}^2$. The complex hyperbolic space $\mathbb{C}\mathbf{H}^2 \subset \mathbb{P}^2$ is the set of negative lines endowed with the distance function

$$\cosh^2\left(\frac{d([z],[w])}{2}\right) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}.$$

Note that we have the inequality

$$\langle z, w\rangle\langle w, z\rangle \geq \langle z, z\rangle\langle w, w\rangle$$

for all negative vectors z and w, with equality if and only if z and w are linearly dependent. Hence the function d defined in 3.1 is well defined. The proof that this defines a metric can be found from [9].

Now define homogeneous coordinates on $\mathbb{P}^2$ by setting the last coordinate equal to 1 by and projecting onto the first two coordinates. This coordinate patch leaves out a subspace isomorphic to $\mathbb{P}^1$, but yields a global chart for $\mathbb{C}\mathbf{H}^2$. Negative vectors correspond to points (z_1, z_2) in the unit ball

$$\mathbb{B}^2 = \left\{(z_1, z_2) \in \mathbb{C}^2 \colon \sqrt{|z_1|^2 + |z_2|^2} < 1\right\}.$$

Denote by $\mathbb{B}^2$ the open unit ball in $\mathbb{C}^2$. We can express the metric above in the ball model by letting the third coordinate equal to 1 as

$$\cosh^2\left(\frac{d_{\mathbb{B}^2}(x, y)}{2}\right) = \frac{(1 - (x, y))(1 - (y, x))}{(1 - |x|^2)(1 - |y|^2)},$$

in terms of the usual euclidean inner product

$$(x, y) = x_1\bar{y}_1 + \ldots + x_n\bar{y}_n.$$

The unit ball equipped with this distance function is called the ball model for complex hyperbolic space.

Up until this point everything has been the same as in the euclidean case. However the properties of the infinitesimal forms of the complex hyperbolic and real hyperbolic metrics are different. Kähler potentials yield a way to defining Riemannian metric tensors on domains of $\mathbb{C}^n$, which are isometric to the complex hyperbolic space. This resulting Riemannian metric is often called the Bergman metric. The detailed construction can be found in [3]. The Bergman metric on the unit ball $\mathbb{B}^2$ is given by

$$g = \frac{2(z_1\bar{z}_2 d\bar{z}_1 dz_2 + \bar{z}_1 z_2 dz_1 d\bar{z}_2 + dz_1 d\bar{z}_1 + d\bar{z}_1 dz_2)}{(1 - \|z\|)^2}.$$

We will simply refer to this metric tensor as the hyperbolic metric. The connection between the Riemannian metric g and the distance function $d_{\mathbb{B}^2}$ is given by the next proposition.

Proposition 3.1. *[3, Ch. 3] The hyperbolic metric tensor g induces the distance function $d_{\mathbb{B}^2}$, ie the metric $d_{\mathbb{B}^2}$ is intrinsic.*

It is apparent that any linear maps preserving the quadratic form are isometries. Maps of this type form a subgroup $U(2,1)$ of $GL(3)$ called the unitary group. It is standard to force the action of the unitary group on the projective space to have a trivial kernel by factoring out the center Z of $U(2,1)$ consisting of scalar matrices $\lambda\mathbb{I}_3$, which produces the projective unitary group $PU(2,1)$. Elements $g \in \mathrm{PU}(2,1)$ are divided into three types in the literature. If g fixes exactly one point in $\partial\mathbb{C}\mathbf{H}^2$, it is called parabolic. These are the type of transformations that are intimately connected with Heisenberg geometry, and we shall explore this connection in the next section. The group $PU(2,1)$, the complex hyperbolic group contains the map induced by complex conjugation in each component.

A *geodesic subset* $N \subset M$ of a length metric space has the property that shortest paths between two points in N are contained in N. This is an analogue of convex sets in euclidean space, where straight line segments are contained in the convex subset. A complex geodesic is the projective image of a two-dimensional subspace of $\mathbb{C}^{2,1}$. A projective line can be defined by a nonzero vector x and a tangent vector $u \in x^{\perp}$, and two complex projective lines always have a unique intersection. In hyperbolic space $\mathbb{C}\mathbf{H}^2$, complex geodesics are geodesic submanifolds [3]. It follows that geodesics in $\mathbb{C}\mathbf{H}^2$ are unique.

We will state one more result from [3], which shows that $\mathbb{C}\mathbf{H}^2$ does have large isometry group.

Proposition 3.2. *[3] The group $PU(2,1)$ of isometries of $\mathbb{C}\mathbf{H}^2$ is transitive, ie $\mathbb{C}\mathbf{H}^2$ is homogenous. Moreover the stabilizer of a point $x \in \mathbb{C}\mathbf{H}^2$ in $PU(2,1)$ is transitive on the set of tangent vectors at x, ie $\mathbb{C}\mathbf{H}^2$ is isotropic.*

Similarly, as in the case of the real hyperbolic manifold, the $\mathbb{R}$-affine lines (of the form $tz + w$, for a real parameter t) in $\mathbb{C}^3$ map to geodesics under the projectivisation map. In particular, geodesics starting from the origin are straight line segments. One way to see this is to notice that the restriction of the hyperbolic metric to a complex geodesic V is simply the real hyperbolic metric on the plane. Hence the geodesics are $\mathbb{R}$-affine lines in the subspace $\mathbb{P}^{-1}V$, and in the whole space $\mathbb{C}^3$.

3.2 The Siegel domain and the embedded Heisenberg group

We can construct the natural extensions of affine maps on the horizontal plane to the whole Heisenberg group by considering a certain stabiliser subgroup of the complex hyperbolic group. The Heisenberg group appears as an embedded subgroup of the complex hyperbolic group. This section is devoted to finding an explicit expression for this subgroup as a matrix subgroup of the projective unitary group $PU(2,1)$. As a bonus, we obtain the group of similarity transformations on the Heisenberg group consisting of transformations, which correspond to euclidean translations, rotations and dilations.

Define the map $W\colon \mathbb{C}^2 \longrightarrow \mathbb{P}^2$ by

$$(z_1, z_2) \mapsto \begin{bmatrix} z_1 \\ \frac{1}{2} - z_2 \\ \frac{1}{2} + z_2 \end{bmatrix}.$$

This map is a well defined linear injective embedding, by which the hyperbolic part $\mathbb{C}\mathbf{H}^2$ corresponds to an unbounded subset $\mathfrak{H} = W^{-1}\mathbb{C}\mathbf{H}^2$ of $\mathbb{C}^2$.

There is exactly one point in the complement of the image of W. We denote this point by p_∞, which is represented in $\mathbb{C}^{2,1}$ by the vector

$$\tilde{p}_\infty = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

The subset $\mathfrak{H}$ is called the Siegel domain. Consider the restriction $C\colon \mathbb{B}^2 \longrightarrow \mathfrak{H}$ of W^{-1}. In homogenous coordinates $C(z_1, z_2) = (w_1, w_2)$ is given by [3]

$$z \in B^2 \longleftrightarrow w \in \mathfrak{H}$$

$$z_1 = \frac{2w_1}{1+2w_2} \quad w_1 = \frac{z_1}{1+z_2}$$

$$z_2 = \frac{1-2w_2}{1+2w_2} \quad w_2 = \frac{1}{2}\frac{1-z_2}{1+z_2}.$$

The coordinate transformations $C\colon \mathbb{B} \longrightarrow \mathfrak{H}$ is will be called the *Cayley tranformation.* The boundary extension defines a unique diffeomorphism $C\colon \mathbb{S}^3\backslash\{(0,-1)\} \longrightarrow \partial\mathfrak{H}$. We will call this map by the same name and denote it by the same symbol. The domain will be made evident from the context. This map will be used in finding algebraic expressions for transformations induced by self-maps of the boundary of the hyperbolic space.

Recall that a defining function of a domain D in $\mathbb{C}^2$ is a smooth function $f\colon \mathbb{C}^2 \longrightarrow \mathbb{R}$ such that the domain corresponds to the subset where $f > 0$. Similarly, a function can be used to define a hypersurface of $\mathbb{C}^2$ by the condition $f = c$ for $c \in \mathbb{R}$. A defining function for the Siegel domain is given by

$$f(w_1, w_2) = 2\operatorname{Re} w_2 - w_1\bar{w}_1. \tag{3.1}$$

Horospheres are submanifolds of $\mathbb{C}\mathbf{H}^2$, which are stable under isometries of the the complex hyperbolic space that fix a point on the boundary. In a certain sense, they are equidistant from the unique fixed point on the boundary. Level sets $f^{-1}(\phi)$ for positive values of ϕ of the defining function f are precisely the horospheres corresponding to the fixed point $\mathbb{P}(p_\infty)$. This simple expression for horospheres is the reason for choosing to work in the Siegel domain model for the complex hyperbolic space, when studying the parabolic subgroup of the complex hyperbolic group.

Next we will explicitly describe a subgroup of the isometry group of $\mathbb{C}\mathbf{H}^2$. Denote the Lie algebra of $U(2,1)$ by $\mathfrak{u}(2,1)$. Let δM denote a matrix of infinitesimal elements. Now the condition that a matrix in the neighbourhood of the identity is in $U(2,1)$ can be written $(I+\delta M)^* J(I+\delta M) = J$, where J is the matrix of the $(2,1)$-indefinite quadratic form. By multiplying and only considering first order terms, we get the condition

$$JM^* + MJ = 0,$$

which is equivalent to M having the form

$$\begin{pmatrix} \mathrm{i}r & a & b \\ -\bar{a} & \mathrm{i}s & c \\ \bar{b} & \bar{c} & \mathrm{i}t \end{pmatrix},$$

where r, s and t are real and a, b and c complex numbers. Now consider the subgroup F of $U(2,1)$, which fixes the element $p_\infty \in \mathbb{P}^2$. The Lie algebra $\mathfrak{f}$ of the group F is the set

$$\mathfrak{f} = \{M \in u(2,1) \colon Mp_\infty = \lambda p_\infty \text{ for some } \lambda \in \mathbb{C}\}.$$

By substituting we get the relations

$$\begin{aligned} b - a &= 0 \\ c - \mathrm{i}s &= -\lambda \\ it - \bar{c} &= \lambda. \end{aligned}$$

From these it is easy to see that an element in $\mathfrak{f}$ is of the form

$$\begin{pmatrix} \mathrm{i}r & a & a \\ -\bar{a} & \mathrm{i}s & u+\mathrm{i}s \\ \bar{a} & u-\mathrm{i}s & -\mathrm{i}s \end{pmatrix},$$

where r, s and u are reals and a is a complex number. A basis for $\mathfrak{f}$ can be given by choosing

$$V_1 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & \mathrm{i} & \mathrm{i} \\ \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{i}/2 & \mathrm{i}/2 \\ 0 & -\mathrm{i}/2 & -\mathrm{i}/2 \end{pmatrix},$$

$$V_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V_5 = \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Define the subalgebras $\mathfrak{f}_1 = \mathrm{span}_{\mathbb{R}}\{V_1, V_2\}$, $\mathfrak{f}_2 = \mathrm{span}_{\mathbb{R}}\{V_3\}$, $\eta = \mathrm{span}_{\mathbb{R}}\{V_4\}$ and $\mathfrak{m} = \mathrm{span}_{\mathbb{R}}\{V_5\}$. In the following, we will see how V_1, V_2 and V_3 span the Heisenberg Lie algebra, whose elements $p = (z, v) \in \mathbb{H}$ shall be identified with 3 by 3 matrices of the form

$$\begin{pmatrix} 0 & z & z \\ -\bar{z} & \mathrm{i}v/2 & \mathrm{i}v/2 \\ \bar{z} & -\mathrm{i}v/2 & -\mathrm{i}v/2 \end{pmatrix}.$$

Because of step two nilpotency of the algebra $\mathfrak{f}_1 \oplus \mathfrak{f}_2$, the exponential map is easy to compute. We obtain the following representation of the Heisenberg group as a group of complex 3-by-3 matrices acting on $\mathbb{C}\mathbf{H}^2$

$$H(z, v) = \begin{pmatrix} 1 & z & z \\ -\bar{z} & 1 - \frac{1}{2}(|z|^2 - \mathrm{i}v/2) & -\frac{1}{2}(|z|^2 - \mathrm{i}v/2) \\ \bar{z} & \frac{1}{2}(|z|^2 - \mathrm{i}v/2) & 1 + \frac{1}{2}(|z|^2 - \mathrm{i}v/2) \end{pmatrix}.$$

We will denote this subgroup of the complex hyperbolic group by $\mathfrak{N}$. Define the evaluation map on each the complex hyperbolic space of $\mathfrak{N}$ on the point

$$\begin{bmatrix} 0 \\ s \\ 1 \end{bmatrix}$$

by setting

$$H(z, t) \mapsto H(z, t) \begin{bmatrix} 0 \\ s \\ 1 \end{bmatrix} = \begin{bmatrix} (1+s)z \\ \frac{1}{2}(2s + (1+s)v\mathrm{i} - (1+s)|z|^2 \\ \frac{1}{2}(2 - (1+s)v\mathrm{i} + (1+s)|z|^2 \end{bmatrix}. \tag{3.2}$$

The evaluation map can be composed with the Cayley transform to define a map on the Siegel domain. The crucial fact is that the evaluation map is a bijection on each horosphere. Thus the Heisenberg group can be identified with any horosphere by the identification

$$(z, t) \longleftrightarrow H(z, t) \begin{bmatrix} 0 \\ s \\ 1 \end{bmatrix}.$$

We can set $s = 1$ to formally obtain coordinates on the complement of a point on the boundary of the hyperbolic space. The subalgebras $\mathfrak{m}$ and η will generate a group, which acts on the horospheres, and by identification, on the Heisenberg group.

The exponential function on $\mathfrak{m}$ generates the matrix group

$$\begin{pmatrix} e^{u\mathrm{i}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

isomorphic to $U(1)$, parametrised by $u \in \mathbb{R}$. Denote elements of this form by A_u. of By writing out the series for matrix elements in e^{V_4}, one can see that the 1-parameter subgroup generated by V_4 consists of matrices of type

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\lambda) & \sinh(\lambda) \\ 0 & \sinh(\lambda) & \cosh(\lambda) \end{pmatrix},$$

for any real number t. The subgroup generated by

$$(A_u)_{u\in\mathbb{R}} = e^{\mathfrak{m}}$$

and

$$(m_\lambda)_{\lambda\in\mathbb{R}} = e^{\eta}$$

is commutative. Furthermore, the subgroup $\mathfrak{N}$ is normal in $G = \mathrm{Stab}(p_\infty)$. This can be seen by a direct computation

$$\begin{pmatrix} e^{-u\mathrm{i}} & 0 & 0 \\ 0 & \cosh(\lambda) & -\sinh(\lambda) \\ 0 & -\sinh(\lambda) & \cosh(\lambda) \end{pmatrix} \begin{pmatrix} 0 & z & z \\ -\bar{z} & 1-q & -q \\ \bar{z} & q & 1+q \end{pmatrix} \begin{pmatrix} e^{u\mathrm{i}} & 0 & 0 \\ 0 & \cosh(\lambda) & \sinh(\lambda) \\ 0 & \sinh(\lambda) & \cosh(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e^{\lambda-\mathrm{i}u}z & e^{\lambda-\mathrm{i}u}z \\ -e^{\lambda+\mathrm{i}u}\bar{z} & 1-\frac{1}{2}(|e^\lambda z|^2 - \mathrm{i}e^{2\lambda}v) & -\frac{1}{2}(|e^\lambda z|^2 - \mathrm{i}e^{2\lambda}v) \\ e^{\lambda+\mathrm{i}u}\bar{z} & \frac{1}{2}(|e^\lambda z|^2 - \mathrm{i}e^{2\lambda}v) & 1+\frac{1}{2}(|e^\lambda z|^2 - \mathrm{i}e^{2\lambda}v) \end{pmatrix} = H(e^\lambda z, e^{2\lambda}v),$$

where we use the notation $q = \frac{1}{2}(|z|^2 - \mathrm{i}v/2)$. Hence we can express the whole group G as an semidirect product $\mathfrak{N} \rtimes U(1) \times \eta$ and every element of the group can be written uniquely as a product $H(\zeta, v)A_u m_\lambda$.

The group G acts on the Siegel domain by the identification $\mathbb{C}\mathbf{H}^2 \leftrightarrow \mathfrak{H}$ given by the Cayley transformation. Moreover, the action of G preserves horospheres, so it defines an action on each set $f^{-1}\phi$ for $\phi > 0$. As we have identified horospheres with the Heisenberg group, we recover a group of transformations on the Heisenberg group. We will call this group the Heisenberg similarity group $\mathrm{Sim}(\mathbb{H})$. Elements of $\mathfrak{N}$ will be called Heisenberg translations, and elements obtained by conjugating an element m_λ or A_u by a Heisenberg translation will be called Heisenberg dilations and Heisenberg rotations. By using the evaluation map given in 3.2, we can derive coordinate expressions in the Siegel domain for all of these maps.

The connection between the exponential coordinates on the Heisenberg group and the group $\mathfrak{N}$ is now made clear by defining horospherical coordinates (also called Heisenberg coordinates) on the Siegel domain by

$$(\zeta, u + \mathrm{i}v) \in \mathbb{C} \times \mathbb{R}_+ \times \mathbb{R},$$

where $z = w_1 \in \mathbb{C}$ and $u + \mathrm{i}v = 2\bar{w}_2 - \|w_1\|^2$, so that u equals the horospherical height function $\phi(w_1, w_2)$. This change of coordinates is a biholomorphism when restricted to the Siegel domain $\mathfrak{H}$. The image of a point (z, u, v) under the action of $H(\zeta, v)$ is

$$(z + \zeta, u, v + 2\,\mathrm{Im}(\zeta, z) + v).$$

Hence under the identification $H(\zeta, v) \leftrightarrow (\zeta, v)$ the action of $\mathfrak{N}$ simply becomes left translation. In particular, Heisenberg translations are isometries of $\mathbb{H}$.

Elements of G act on the boundary of the complex hyperbolic space by unique extension by continuity. The action of G is formally the same in Heisenberg coordinates for any value of the horospherical height coordinate u and this holds for the boundary extension ($u = 0$) as well.

The following diagram conveniently captures the identifications that are made throughout.

$$\begin{array}{ccccc} \mathbb{H} & \xleftrightarrow{\text{Action of } \mathfrak{N} \text{ on}(0,0)} & \partial\mathfrak{H} & \xleftrightarrow{\text{Cayley map}} & \mathbb{S}^3\backslash\{(0,-1)\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H} & \xleftrightarrow{\text{Action of } \mathfrak{N} \text{ on}(0,0)} & \partial\mathfrak{H} & \xleftrightarrow{\text{Cayley map}} & \mathbb{S}^3\backslash\{(0,-1)\} \end{array} \tag{3.3}$$

Given the above horizontal identifications, an induced map is the unique choice of vertical map, which makes the above diagram commute. This map is given by conjugation by the correct horizontal maps. We will find the coordinate expression for all elements of G in section 3.5.

3.3 Left-invariant Riemannian metrics on the Heisenberg group

A natural way to approximate the CC geometry of $\mathbb{H}$ is by considering a family of left invariant Riemannian metrics orthogonal with regard to the basis vectors X, Y and ν such that the vertical direction is given a large weight L. As the parameter L tends to infinity the metrix structure $\mathbb{H}$ gradually tends to the CC metric structure in a strong Gromov-Hausdorff sense. Limiting arguments can then be used to prove theorems about the topological and metric structure of the sub-Riemannian Heisenberg group. We will approach the approximation of the

CC metric by embedding the Heisenberg group into the larger complex hyperbolic space and by considering restrictions of the hyperbolic metric.

The Bergman metric on the Siegel domain is given by the expression [3]

$$\frac{4}{f(w_1,w_2)^2}\left(dw_2 d\bar{w}_2 - (w_2+\bar{w}_2)\, dw_1 d\bar{w}_1 + \|w_1\| dw_1 d\bar{w}_1 + f(w_1,w_2) dw_1 d\bar{w}_1\right).$$

Substituting the expressions for horospherical coordinates, and restricting to the horosphere $f^{-1}\phi$, we get the expression

$$\frac{4}{\phi^2}\left\{\frac{(d|\zeta|^2 - idv)(d|\zeta|^2+idv)}{4} + |\zeta|^2|d\zeta|^2 - \frac{\zeta d\bar{\zeta}(d|\zeta|^2 - \mathrm{i}dv) + \bar{\zeta}d\zeta(d|\zeta|^2+idv)}{2}\right\}$$

$$+\frac{4}{\phi}|d\zeta|^2$$

Here we write $|\zeta|^2$ for $\zeta\bar{\zeta}$. The outer derivative of this evaluates to $\bar{\zeta}d\zeta + \zeta d\bar{\zeta}$. We also write $|dw|^2$ for $dwd\bar{w}$. Restriction to a horosphere means that we are evaluating vectors orthogonal to $\frac{\partial}{\partial u}$, which are in the kernel of du, so summands containing du as a symmetric factor can be omitted.

When restricted to horospheres, the above expression simplifies down to

$$\frac{1}{\phi}\left\{\frac{1}{\phi}\left(2|\zeta|^2 d|\zeta|^2 - \zeta^2 d\bar{\zeta}^2 - \bar{\zeta}^2 d\zeta^2 + dv^2 - 2\mathrm{i}dv\left(\zeta d\bar{\zeta} - \bar{\zeta}d\zeta\right)\right) + 4|d\zeta|^2\right\},$$

which can be written

$$\frac{1}{\phi}\left\{\frac{1}{\phi}\left(dv - 2\,\mathrm{Im}\,\zeta d\bar{\zeta}\right)^2 + 4|d\zeta|^2\right\} = \frac{1}{\phi}\left\{\frac{1}{\phi}\left(dv + 2(xdy - ydx)\right)^2 + 4(dx^2+dy^2)\right\}.$$

We see that the Heisenberg contact form $\omega = dv - 2\,\mathrm{Im}\,\zeta d\bar{\zeta} = dv + 2xdy - 2ydx$ appears in the expression, and so we write the above in the final form

$$g_\phi = \frac{1}{\phi}\left\{\frac{\omega^2}{\phi} + 4(dx^2+dy^2)\right\}.$$

The matrix form of the above at the origin $x=y=v=0$ is

$$\frac{1}{\phi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\phi \end{pmatrix}.$$

We see that the system $\sqrt{\phi}\frac{\partial}{\partial x}, \sqrt{\phi}\frac{\partial}{\partial y}, \phi\frac{\partial}{\partial v}$ is orthonormal at the origin. As in 2.6, the unique translation invariant frame is given by

$$U_1 = \sqrt{\phi}\left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial v}\right),\quad U_2 = \sqrt{\phi}\left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial v}\right),\quad U_3 = \phi\frac{\partial}{\partial v}.$$

This system is orthonormal at each point, as Heisenberg translations are isometries of the Bergman metric and also of its restriction g_ϕ. Let γ be a C^1 curve given by control functions a_1, a_2 and a_3. In other words γ satisfies the equation

$$\gamma' = a_1 U_1 + a_2 U_2 + a_3 U_3.$$

Upon rescaling by a factor ϕ, the length of γ is given by

$$\begin{aligned} L_{g_\phi}(\gamma) &= \int \sqrt{g_\phi(\gamma', \gamma')} \\ &= \int \sqrt{a_1^2 + a_2^2 + \phi a_3^2}. \end{aligned}$$

This Riemannian length structure gives rise to a distance function d_ϕ.

The quadratic form in the integral converges to the quadratic form related to the CC metric defined in section 2.1. The metric spaces $(\mathbb{H}, d_\phi)$ converge in the Gromov-Hausdorff sense to the space $(\mathbb{H}, d_C)$. This implies, for instance, that length minimising horizontal curves in $(\mathbb{H}, d_C)$ are aquired as uniform limits of geodesic arcs in $(\mathbb{H}, d_\phi)$. Also the quantity

$$\limsup_{\phi \to 0} d_C(x, y) - d_\phi(x, y)$$

in any compact subset is zero as ϕ tends to zero. The details and proofs of results related to Gromov-Hausdorff convergence are found in [8].

We can use the Riemannian approximation to derive consequences of propositions 2.5 and 2.6. We will only use completeness of the Riemannian approximants, and not the explicit forms of the geodesics or the distance function.

First of all, a distance realising path on γ_{d_C} can be constructed by finding a family of length minimising geodesics on the Riemannian manifolds $(\mathbb{H}, g_\phi)$. Their existence follows from the metric completeness of the manifolds $(\mathbb{H}, g_\phi)$ and the Hopf-Rinow theorem (Proposition 2.5). Hence, length minimising curves always exist between all pairs of points on the sub-Riemannian manifold $(\mathbb{H}, d_C)$.

Next we wish to construct an infinite extension for a geodesic starting from $p \in \mathbb{H}$. Let γ be a geodesic between two points p and q on $\mathbb{H}_{d_C}$. Consider the sequence γ_ϕ of paths converging uniformly to the path γ, such that the paths $\gamma_\phi \colon p \curvearrowright q$ are geodesics on Riemannin manifolds $(\mathbb{H}, g_\phi)$. Every one of these geodesics is uniquely infinitely extendable by Proposition 2.5. In any compact set, these extensions converge uniformly to an extension of the path γ. Hence we have constructed an extension of arbitrary finite length. Consider another extension $\bar{\gamma}$ of the path γ. The path $\bar{\gamma}$ is aquired as a limit of a sequence of curves $\bar{\gamma}_\phi$ on Riemannin manifolds

$(\mathbb{H}, g_\phi)$, whose restrictions will converge to the path γ. The result follows from unique extendability of paths $\bar{\gamma}_\phi$.

Besides these topological properties, which are at this point rather trivial, the Riemannian approximation has other immediate corollaries.

Proposition 3.3. *The Heisenberg group can not be isometrically embedded in euclidean space $\mathbb{R}^n$ for any n.*

Proof. The scalar curvature of the Riemannian approximants diverges everywhere as ϕ tends to zero [8], so the distance function d_C can not be given by a Riemannian metric. $\square$

3.4 CR structures

This section will be devoted to describing a natural structure on three dimensional submanifolds (hypersurfaces) of complex space $\mathbb{C}^2$ induced by the complex structure of the ambient manifold. These so-called CR structures will turn out to yield contact structures for 3-manifolds for a large class of 3-manifolds that can be embedded into the space $\mathbb{C}^2$. Instead of proving the general case, we will check the contact criterion in each case. We will begin with a quick outline of notation.

Earlier, we used complex notation formally without reference to real or imaginary parts. The complex structure of the space $\mathbb{C}^2$ can be described in terms of real differential geometry as follows. We can relate the space $\mathbb{R}^4$ and $\mathbb{C}^2$ with the identification $z_i = x_i + \mathrm{i}y_i$, for $i \in \{1, 2\}$. At the level of tangent vectors, the correct identification is given by

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \mathrm{i}\frac{\partial}{\partial y_i}\right),$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \mathrm{i}\frac{\partial}{\partial y_i}\right).$$

These relations can be justified by the conditions $\frac{\partial}{\partial z}z = 1$ and $\frac{\partial}{\partial \bar{z}}\bar{z} = 1$. One can spot the connection to holomorphic functions, which are the functions annihilated by $\frac{\partial}{\partial \bar{z}}$.

The standard complex structure on $\mathbb{R}^4$ is the linear map $\mathbb{J}\colon T\mathbb{C}^2 \longrightarrow T\mathbb{C}^2$ defined by the matrix

$$\mathbb{J} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let W be a real hypersurface of $\mathbb{C}^2$ and let $f\colon W \longrightarrow \mathbb{R}$ be a smooth defining function for W (ie $W = f^{-1}0$ and 0 is a regular value for f). A *CR structure* for W is given as the intersection

$$TW \cap \mathbb{J}TW.$$

These subbundles of the tangent bundle $T\mathbb{C}^2$ can be given in terms of the defining function as

$$TW = \mathrm{Ker}(df)$$

$$\mathbb{J}TW = \mathrm{Ker}(df \circ \mathbb{J}).$$

Let X be a vector satisfying the above requirements. Write X in terms of the real basis as

$$X = a_1\frac{\partial}{\partial x_1} + b_1\frac{\partial}{\partial y_1} + a_2\frac{\partial}{\partial x_2} + b_2\frac{\partial}{\partial y_2}.$$

Substituting the above conditions yields

$$\frac{\partial f}{\partial x_1}a_1 + \frac{\partial f}{\partial y_1}b_1 + \frac{\partial f}{\partial x_2}a_2 + \frac{\partial f}{\partial y_2}b_2 = 0$$

and

$$-\frac{\partial f}{\partial x_1}b_1 + \frac{\partial f}{\partial y_1}a_1 - \frac{\partial f}{\partial x_2}b_2 + \frac{\partial f}{\partial y_2}a_2 = 0.$$

Finally changing to the complex basis defined above leads to the equivalent condition

$$\left(\frac{\partial f}{\partial \bar{z}_1}d\bar{z}_1 + \frac{\partial f}{\partial \bar{z}_2}d\bar{z}_2 - \frac{\partial f}{\partial z_1}dz_1 - \frac{\partial f}{\partial z_2}dz_2\right)(X) = 0$$

Thus we define the calibrating form σ for the CR structure can now be given in terms of the complex coordinates by

$$\sigma = \frac{\partial f}{\partial \bar{z}_1}d\bar{z}_1 + \frac{\partial f}{\partial \bar{z}_2}d\bar{z}_2 - \frac{\partial f}{\partial z_1}dz_1 - \frac{\partial f}{\partial z_2}dz_2.$$

There is a proper generalisation of this definition for CR structures (see [3]), which yields analogous structures for $2n-1$-dimensional real manifolds that cannot be imbedded into complex n-space.

We will identify the CR structure with the horizontal structure, as we will next show that the horizontal structures of the embedded Heisenberg groups will converge to the CR structure at the limit. Let ϕ be a positive real number. Substituting the defining function $2\mathrm{Re}w_2 - |w_1|^2 = \phi$ for the horosphere $f^{-1}(\phi)$ on the Siegel domain yields the contact form

$$2\mathrm{i}\,\mathrm{Im}(\bar{w}_1 dw_1 + d\bar{w}_2).$$

We might as well rescale by 2i to get the form

$$\tau = \mathrm{Im}(\bar{w_1} dw_1 + d\bar{w}_2).$$

on the horosphere $f^{-1}(\phi)$. In Heisenberg coordinates (ζ, v) for horospheres defined by

$$w_1 = \zeta.$$

and

$$w_2 = \frac{1}{2}(u + |\zeta|^2 - iv),$$

the form τ becomes the contact form on Heisenberg group

$$\omega = \mathrm{Im}(\bar{\zeta} d\zeta) + dv = dv + 2(xdy - ydx), \tag{3.4}$$

where $\zeta = x + \mathrm{i}y$. Hence we have seen that the CR structure on horospheres of the Siegel domain corresponds, in Heisenberg coordinates, exactly to the horizontal structure of the Heisenberg group. The expression does not depend on horospherical height. We can define a contact form on the boundary of $\mathfrak{H}$ simply by setting the horospherical height to zero.

A similar computation on the unit sphere in $\mathbb{C}^2$ yields the form

$$\tau = \bar{z}_1 dz_1 - z_1 d\bar{z}_1 + \bar{z}_2 dz_2 - z_2 d\bar{z}_2.$$

The geometry of any space is best described by the classes mappings, which preserve particular geometric features. In the sub-Riemannian setting, a natural class of maps is formed by maps which preserve the contact structure. This property can be characterised in the following way. A diffeomorphism between two sub-Riemannian manifolds (M, ω) and (N, η), which satisfies the equation

$$f^*\omega = g\eta \tag{3.5}$$

for some smooth, everywhere nonzero function g, is called a contact transformation, or more abstractly a morphism in the category of sub-Riemannian manifolds or a contactomorphism. Self-maps which are contact transformations are the starting point for the study of quasiconformal and quasiregular mappings on the Heisenberg group.

The Cayley map is an example of a contactomorphism, which can be seen directly by substituting the Cayley transform to the contact form τ.

Proposition 3.4. *The Cayley transform $C\colon \mathbb{S}^3 \backslash \{(0, -1)\} \longrightarrow \partial\mathfrak{H}$ is a contactomorphism.*

Hence the Cayley transformation identifies the horizontal structures on the boundary of the Siegel domain and the unit sphere $\mathbb{S}^3$. We will see in section 4.2 how Cayley map identifies the infinitesimal metric structure of the Heisenberg group with a metric version of the asymptotic boundary $\mathbb{S}^3$ of the complex hyperbolic space.

Contactomorphisms do not in general preserve the metric structure: The CC quadratic form can be distorted arbitrarily by a mapping generated by a linear mapping of the basis for the horizontal tangent space at the identity. Linear stretching of tangent spaces is intimately connected with metric distortion. If the topology of the manifold allows, a contact automorphism may smoothly distort the bilinear form, which defines the CC metric on sub-Riemannian manifold, while contactomorphisms between two sub-Riemannian manifolds M and N may smoothly distort the function $M \longrightarrow N$ defined by $q \mapsto d_C(p,q)$. The characterisation theorem on conformal maps on the Heisenberg group in section 3.6 is proved by bounding the metric or the conformal distortions. We will also give an example of a family of maps, which distort the metric structure in a bounded manner.

3.5 The compactified Heisenberg group

We now consider the Heisenberg group as a subgroup $\mathfrak{N}$ of G acting 1-transitively on the punctured unit sphere.

Let $\hat{\mathbb{H}}$ denote the one-point compactification of $\mathbb{H}$ given by the inclusion map $\mathbb{S}^3 \backslash \{0,1\} \longrightarrow \mathbb{S}^3$. This map is a contact map by Proposition 3.4, when we endow the sphere $\mathbb{S}$ with its standard contact structure as described in section 3.4. The explicit compactification yields a way of formulating questions about geometric objects extending to infinity. This question of infinite extension was already considered in Proposition 2.6, but now we can specialise the question to particular properties of specific objects. We return to the algebraic representation of geometric objects extended to infinity in section 4.1.

The explicit coordinates are obtained as follows. The point (z,v) is identified with the point $(z, u+\mathrm{i}v)$ on each horosphere. The coordinate transformation from horospherical coordinates to euclidean coordinates of the Siegel domain extend to the boundary uniquely, so at the limit we get the correspondence

$$(z,v) \longleftrightarrow (z, \frac{1}{2}(|z| - \mathrm{i}v)).$$

The Cayley map is a diffeomorphism on the punctured unit sphere. Using these identifications, the point $(z,v)\in\mathbb{H}$ is taken to the point

$$(z,v)\longleftrightarrow\left(\frac{2z}{1+|z|^2-\mathrm{i}v},\frac{1-|z|^2+\mathrm{i}v}{1+|z|^2-\mathrm{i}v}\right).$$

The inverse map, which is well defined for $z_2\neq -1$, is given by

$$(z_1,z_2)\longleftrightarrow\left(\frac{z_1}{1+z_2},-\operatorname{Im}\frac{1-z_2}{1+z_2}\right).$$

In Heisenberg coordinates the expressions for the action of A_u and m_λ on a point $p=(z,v)$ is given by

$$A_u(z,v)=(e^{u\mathrm{i}}z,v)$$

and

$$m_\lambda(z,v)=(e^{-\lambda}z,e^{-2\lambda}v).$$

for $\lambda\in\mathbb{R}^+$ and $t\in\mathbb{R}$. These parametrisations have the handy Lie group homomorphism properties

$$A_u\circ A_{u'}=A_{u+u'}$$

and

$$m_\lambda\circ m_{\lambda'}=m_{\lambda+\lambda'},$$

but for our purposes it makes sense to reparametrise the dilations m_λ by replacing λ with $-\log\lambda$. Denote this reparametrised version by M_λ. It is given by

$$M_\lambda(z,v)=(\lambda z,\lambda^2 v),$$

where the parameter λ ranges over the positive reals. We now recover the classical dilation equation

$$d_{\mathbb{H}}(0,M_\lambda p)=\lambda d_{\mathbb{H}}(0,p)$$

in the space $(\mathbb{H},d_{\mathbb{H}})$ for all points $p\in\mathbb{H}$. We will use the reparametrised version Heisenberg dilations in the following.

We return to the question of bilipschitz-equivalence of the two metrics $d_{\mathbb{H}}$ and d_C to give a proof using the Heisenberg dilation. This is not a proof for the tight lower bound. The proof depends first of all on the continuity of $d_C(0,p)$ in the euclidean topology. A. Bellaïche provides a proof in [2] by comparing the CC metric of the matrix model (see [8]) of the Heisenberg group to the metric $1/3(|x|+|y|+|z|^{\frac{1}{2}}$, which induces the euclidean topology. The matrix model is obtained via a diffeomorphism given in [8] from the exponential model we use. This implies the continuity of d_C with respect to the euclidean distance and the Heisenberg metric, as these two metrics clearly induce the same topology.

Armed with the Heisenberg dilation and the continuity of d_C, we finally prove the following essential theorem.

Proposition 3.5. *The two metrics $d_{\mathbb{H}}$ and d_C are bilipschitz-equivalent.*

Proof. Consider the CC metric unit ball $\mathbb{B}_C(0,1)$. As the Heisenberg metric induces the same topology as the CC metric, we can choose a ball $\mathbb{B}_{\mathbb{H}}(0,r)$ in the Heisenberg metric that is entirely contained in $\mathbb{B}_C(0,1)$. By applying the dilation property, it is clear that the rescaled ball $\mathbb{B}_C(0,\lambda r)$ is contained in the Heisenberg metric $\mathbb{B}_C(0,\lambda)$ for any $\lambda \in \mathbb{R}$. This proves that we can use r as the lipschitz constant for the lower bound in equation 2.5. □

Heisenberg rotations reveal the rotational symmetry present in both the metric geometry of $(\mathbb{H}, d_{\mathbb{H}})$ (where it is rather obvious from the expression of the metric) and the sub-Riemannian manifold $\mathbb{H}$. In the CC context, the rotations could have been realised through an orthonormal change of basis for the horizontal structure at the origin. The following proposition follows immediately.

Proposition 3.6. *Heisenberg rotations and translations are isometries of the Heisenberg group.*

This fact is also clear from the symmetry of the parametrisations for geodesics. We also have the following scaling property for Heisenberg dilations in terms of the CC metric.

Proposition 3.7. *The (reparametrised) Heisenberg dilation satisfies*

$$d_C(0, M_\lambda p) = \lambda d_C(0,p)$$

for any $0, p \in \mathbb{H}$.

Proof. The tangent map of M_λ is given by

$$(M_\lambda)_* \gamma' = (s\gamma_1, s\gamma_2, s^2\gamma_3) \tag{3.6}$$

for any horizontal curve γ, so the dilated curve $M_\lambda \circ \gamma$ is a horizontal. The action on the length is given by

$$L(M_\lambda \circ \gamma) = \lambda L(\gamma),$$

which proves the claim. □

The infinity point $(0,-1)$ is a fixed point of every element of G.

The identification of the contact structure on $\mathbb{S}^3$ and $\mathbb{H}$ provides new information about the geometry of $\mathbb{H}$. Consider the inversion $j\colon \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ defined by the equation

$$(z_1, z_2) \mapsto (-z_1, -z_2).$$

This map can be aquired as the boundary extension of the isometry of the complex hyperbolic space defined on the ball model by the same equation. Moreover, the map preserves the contact form on $\mathbb{S}^3$ defined in equation 3.4, so it is a contact transformation. The map j induces a map on the Heisenberg group, and the formula

$$j(z,v) = \left(\frac{-z}{|z|^2 + \mathrm{i}v}, \frac{-v}{|z|^4 + v^2} \right). \tag{3.7}$$

can be read from the diagram in equation 3.3. The inversion is a contact transformation on the Heisenberg group by composition. Conjugation by translations and dilations yields the group of general Heisenberg inversions, which are contact transformations.

If we look at the coordinate expressions of the maps $\mathrm{Sim}(\mathbb{H})$ and the Heisenberg inversion, we see that these maps restrict to the usual Möbius maps on the the $v = 0$ plane, and in fact any horizontal plane.

3.6 Conformal Geometry of the Heisenberg Group

We continue the study of contact mappings in a class with restricted infinitesimal metric distortion. This section contains a brief introduction to the theory of quasiconformal mappings, with a view towards fully describing the conformal group of the sub-Riemannian Heisenberg group. We include a series of proposition due to the paper [1] by Korányi and Reimann. Our goal will be the statement of their result on the rigidity of the conformal structure of the sub-Riemannian Heisenberg group.

Intuitively, a conformal map is a map, which stretches the metric uniformly in each direction. In order to follow [1], we relax the requirement of smoothness to C^1 differentiability.

Define the quantities

$$a(p,r) = \max\{|f(x) - f(p)|\colon |x,p|_{\mathbb{H}} = r\}$$

and

$$b(p,r) = \min\{|f(x) - f(p)|\colon |x,p|_{\mathbb{H}} = r\},$$

where $\mathbb{B}$ denotes the metric ball of the Heisenberg metric $d_{\mathbb{H}}$. Let K be a real number such that $K \geq 1$. The C^2-diffeomorphism f is K-*quasiconformal* at p if the ratio, or the conformal distortion at point p, satisfies the inequality

$$\limsup_{r \to 0} \frac{a(p,r)}{b(p,r)} \leq K.$$

A K-quasiconformal map with $K = 1$ is called *conformal.* One can use the CC-metric to define quasiconformality. Due to the bilipschitz-equivalence of the two metrics, they lead to the same classes of quasiconformal maps.

The first aim is to relate K-quasiconformality to the differential and contactness of map. In this vein we cite the following theorems.

Proposition 3.8 ([1])**.** *A differentiable K-quasiconformal mapping with a non-singular derivative is a contact transformation.* □

Define the notation

$$\lambda_1 = \sup_{|V|_H=1} |f_* V|_H, \quad \lambda_2 = \inf_{|V|_H=1} |f_* V|_H,$$

where the length of the vector $V = aX + bY + cV$ is measured from the CC inner product $\sqrt{a^2 + b^2}$.

Proposition 3.9. *[1] A contact transformation which is twice differentiable and satisfies*

$$\frac{\lambda_1}{\lambda_2}(p) \leq K$$

is K-quasiconformal. □

Proposition 3.10. *Elements of the similarity group* $\mathrm{Sim}(\mathbb{H})$ *are contact transformations and they satisfy $\lambda_1 = \lambda_2$.*

Proof. Contactness of a map is preserved by composition and in inverting a map. Isometries are clearly contact transformations, so it suffices to consider the dilation about a vector about the origin, for which contactness follows from the expression for the tangent map in 3.6. The second part follows from the fact that the tangent maps of elements of G are orthogonal on horizontal bundle. □

As a corollary to Proposition 3.10 we have the following theorem.

Corollary 3.1. *[1] Elements of G are conformal.* □

The conformality of the Heisenberg inversion will follow from our final result, Corollary 4.1.

Finally we state the converse, which is the main result of [1].

Proposition 3.11. *[1] The full conformal group is generated by the Heisenberg similarity group and the Heisenberg inversion $j_{\mathbb{H}}$.* □

4 Visual Geometry

4.1 Visual geometry of the Heisenberg group

We now return to performing computations in exponential coordinates. In particular, we focus on the parametrisations for geodesic arcs derived section 2.4. The focus is shifted from studying the space itself to studying a pointed space of geodesic arcs and their infinite extensions called rays. We will title this shifted viewpoint as visual geometry. We will define a parametrisation of the Heisenberg group in terms of these geodesics and study its topological properties. The resulting map is as close as one can get to defining an exponential map on the sub-Riemannian Heisenberg group.

A geodesic with the initial point p, which has been extended to infinity will be called a *ray* emanating from p. In the Heisenberg space $\mathbb{H}$ all geodesic arcs can be infinitely extended into rays by the parametrisation given in section 2.4. An example of a geodesic ray emanating from the origin in $\mathbb{H}$ is given by the lift of the curve $s \mapsto e^{i}s - 1$, where $s \in [0, \infty)$.

The circle bundle $H/\mathbb{R}^* \simeq S^1\mathbb{H}$ can be given a global coordinate $t \in \mathbb{S}^1$, by setting

$$t \mapsto \cos(t)X + \sin(t)Y.$$

Recall that we parametrised the set of nonconstant geodesic arcs starting from the origin in terms of triplets $(t, c, s) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}^+$. The collection of geodesic arcs is carried to any point p by left translation. We will assign the symbol GA_p for the space of geodesic arcs staring from p, where the value 0 for the length parameter corresponding to the trivial geodesic is included for convenience. These triplets correspond to the choice of direction, curvature and length of the geodesic. From now on, we identify triplets of this form with the unit speed parametrised geodesic arc starting from a fixed point $p \in \mathbb{H}$. We will assign the symbol GA_p to this space of geodesic arcs.

Let q be a point on $\mathbb{H}$ and (t, c, s) be a geodesic arc starting from p, which we will now write explicitly. Heisenberg translations, rotations and dilations act on the geodesics according to the expressions

$$L_q(t, c, s)_p = (t, c, s)_{qp}\,,$$

$$M_\lambda(t, c, s)_p = \left(t, \frac{c}{\lambda}, \lambda s\right)_{M_\lambda p}$$

and

$$A_u(t, c, p)_p = (t', c, s)_{A_u p}\,,$$

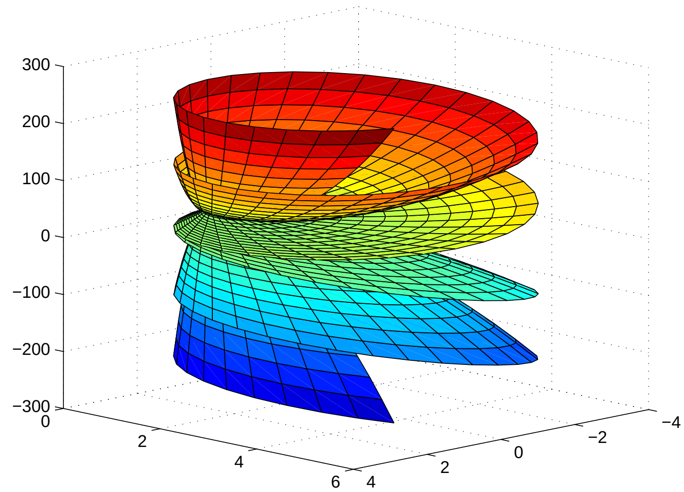

FIGURE 2: *The view from the origin in the $t = 0$ direction.*

where t' is the global coordinate obtained by left translating the vector

$$\cos(t+u)\frac{\partial}{\partial x} + \sin(t+u)\frac{\partial}{\partial y}.$$

This is easily seen to define the action of G on the bundle GA$\mathbb{H}$ of geodesic arcs.

Example 4.1. While the horizontal plane appears two dimensional, it is still a way to make a two-dimensional visual picture of ones surroundings, based on differentiating rays by their curvature. Natural questions of visibility can be asked in CC-geometry: How do the shape and the position of an object determine its appearance to an observer? The answer depend on the extrinsic features of the coordinate system, as well as the intrinsic features of the geometry. The notion of visibility is copied from euclidean intuition: A subset is visible from the point p in the direction of a ray, if the ray crosses the subset.

To get a feeling of the non-isotropic nature of visual CC-geometry, consider the following example. How does the appearance of a sphere of radius L depend on its location? If the centre of the sphere is located on a ray of zero curvatur and is far away from the observer, the image resembles a rounded box of width proportional to L and height proportional to $\sqrt{L}$. As the sphere is moved closer to the observer, portions of the sphere, which are off the horizontal plane, begin appearing at areas of larger absolute value of curvature.

Next assume that the center of the sphere is vertical relative to the observer. Firstly, the image is homogenous in the t-direction and fills the whole visual cylider for large values of c. The observer will also see horizontal strips, whose number and width depend on the distance and radius of the sphere. Notice how the visual image is invariant under Heisenberg similarities. It is instructive to try to figure out what features of the space in this highly coordinate dependent picture are invariant under metric distorting contactomorphisms.

The parametrisation of geodesic arcs set gives us coordinates, in a sense, in terms of direction, curvature and distance by a (noninjective) mapping

$$F_p \colon \mathrm{GA}_p \longrightarrow \mathbb{R}^3,$$

where the image of F_p is defined the be the end point of the geodesic arc (t, c, d) starting from p. This map is a restriction of the end-point map, and its domain is a four dimensional submanifold of the Banach space of functions. Hence the properties of F_p can be described in the language of finite dimensional smooth manifolds. Its expression for $p = 0$ is given by the parametrisation

$$(t, c, s) \longleftrightarrow \left(\frac{1}{c}(\left(e^{\mathrm{ci}(s-\pi/2)} \pm \mathrm{ie}^{\mathrm{it}}\right), \pm\frac{2}{c^2}\left(\sin(cs) - cs\right) \right)$$

for $c \neq 0$ and

$$(t, c, s) \longleftrightarrow \left(se^{\mathrm{it}}, 0\right)$$

All properties of the mapping F_p we are interested in are preserved, when we compose F_p with a left-translation, hence we may assume that $p = 0$.

A metric on the space of geodesic arcs can given by the obvious inclusion of $\mathbb{S}^1 \times \mathbb{R} \times [0, \infty)$ into $\mathbb{R}^4$ as the complement of an open cylinder. The CC length function takes now a particularly simple form. Namely it is given by the projection to the third coordinate, so in particular, it is a continuous map.

We would like F_0 to be a local homeomorphism, which would yield us geodesic coordinates analoguous to an exponential map defined on a manifold. However, as we discussed eariler, geodesics connecting points in vertical relative position are not locally unique. The map F_0 is continuous, as it is a composition of smooth functions when $c \neq 0$, and the limits agree when c approaches zero. Notice, however, that the derivative map $(F_0)_*$ of F_0 does not converge as c tends to 0. Restricting F_0 to nonzero c and the parameter s to values not multiple of $2\pi R$, the restriction of F_0 becomes a smooth covering map onto its image. Allowing c to take the value 0, we get still get a topological covering. Still, the nonimmersivity in the vertical direction can not be circumvented.

Consider the quotient space

$$V(\mathbb{H}) = \mathrm{GA}_p/F_p,$$

where we identifies points with the same images under F_p. For the definition to make much sense, the map F_p should have nice topological properties. On any n-manifold, the above construction always yields locally the space $\mathbb{R}^n$, as one can always find a neighbourhood where the exponential function, which now takes the place of F_p, is a diffeomorphism. The following problem relates the visual geometry of the sub-Riemannian Heisenberg group to the space of geodesic arcs.

Problem 1. Describe the topology of the quotient space $V(\mathbb{H})$.

The answer is mostly given by our previous discussion of the mapping F_p. The lack of local injectivity complicates the situation.

As a simple example of how the viewpoint presented in this section could be applied, we construct arbitrarily long geodesics connecting two points arbitrarily close to each other.

Example 4.2. Let γ_k be the infinite collection of geodesics of increasing distinct lengths between points 0 and $(0, 0, 1)$ as in example 2.2. Apply the Heisenberg dilation about the origin by a factor $1/\sqrt{k}$ on each of the paths, whose indices are perfect squares. We obtain a sequence $\tilde{\gamma}_{k^2}$ such that $\tilde{\gamma}_{k^2}$ connects the points $(0, 0, 0)$ and $(0, 0, k^{-1})$. The CC length of the paths we have constructed satisfies

$$L(\tilde{\gamma}_{k^2}) = \sqrt{\frac{k}{\pi}},$$

which can be made arbitrarily large.

The geodesics $\tilde{\gamma}_{k^2}$ can be extended slightly to construct arbitrarily long geodesics connecting two points, which are not in vertical position relative to each other. Here we use the continuity of CC distance.

Explicit description of the geometry of $\mathbb{H}$ done in this section, as well as sections 2.4 and 3.2 have demonstrated how the sub-Riemannian Heisenberg geometry has the following properties, when restricted to the horizontal plane.

- It is flat in the sense that geodesic triangles with two points on two zero-curvature rays are scaled as in euclidean geometry.
- It is isotropic in the sense that the horizontal plane can be rotated by isometries.
- It the same linear scaling property as the euclidean plane.

Hence, as advocated by A. Korányi in [6], the sub-Riemannian geometry of the Heisenberg group is natural generalisation of two-dimensional flat, homogenous and isotropic geometry, that is, the geometry of the euclidean plane. We have not discussed higher dimensional analogues of the Heisenberg group, but they generalise the geometry of any even dimensional euclidean space in a similar fashion.

As an application of the explicit compactification of the Heisenberg group in section 3.5, we compute the tangent vector of a ray at infinity.

Example 4.3. Consider the ray $(t, 0, 0)$ (zero curvature, emanating from the origin) on the compactified Heisenberg group. In exponential coordinates, this geodesic has the parametrisation depending on the direction $t \leftrightarrow e^{\phi \mathrm{i}}$

$$(e^{\phi \mathrm{i}} s, 0)$$

in terms of the real parameter s.

This geodesic is parametrised on the boundary of the Siegel domain by

$$w_1 = e^{\phi \mathrm{i}} s, \quad w_2 = \frac{s^2}{2}.$$

The Cayley transform takes points of this form to points

$$z_1 = \frac{2e^{\phi \mathrm{i}} s}{1 + s^2}, \quad z_2 = \frac{1 - s^2}{1 + s^2}$$

on the unit sphere $\mathbb{S}^3$. The tangent vector of this curve is

$$\frac{1}{(1 + s^2)^2} \left(2e^{\phi \mathrm{i}}(1 + s^2), -4s\right).$$

The initial and limiting directions of the tangent vector are given by $\left(e^{\phi \mathrm{i}}, 0\right)$ and $\left(-e^{\phi \mathrm{i}}, 0\right)$, respectively, so the direction is reversed in the ambient space $\mathbb{C}^2$. This is what one might guess based on intuition from two-dimensional euclidean geometry.

It may be interesting to know what happens to curved geodesics, and write out a full expression for a mapping from the bundle of rays to the unit tangent space at infinity, where a ray is mapped to its tangent vector at the limit, if one exists.

We may now apply the terminology to the following questions about the existence of maps of bounded distortion, which change the curvature of rays.

Problem 2. Is there a K-quasiconformal map of $\mathbb{H}$ taking a $c > 0$-type geodesic ray to one of type $c = 0$?

According to [10], the problem is related to a in mapping problem complex geometry of exchanging the real and complex slices at the boundary of the complex hyperbolic plane. One can relax the injectivity requirement and ask the question same question in the class of quasiregular mappings.

Problem 3. Is there a K-quasiregular map of $\mathbb{H}$ taking a $c > 0$-type geodesic ray to one of type $c = 0$?

4.2 The Gromov ideal boundary of the complex hyperbolic space

We have considered the compactified Heisenberg space given by the Cayley map. The metric structure on the boundary was aquired as a limit of rescaled restrictions of the Bergman metric in section 3.3, where it was shown that the metric space $(\mathbb{H}, d_C)$ appears as the asymptotic boundary of the complex hyperbolic space. The conformal geometry of the Heisenberg group related to a visual geometry of the complex hyperbolic space, which will be the subject of this section. The connection appears as we define a metric analoguous to the chordal metric, which yields the conformal structure of euclidean space for the unit sphere [15].

The aim is to define a metric on the asymptotic boundary of complex hyperbolic space $\mathbb{C}\mathbf{H}^2$. The boundary is usually formally be defined as a maximal set of metrically (in the Gromov-Hausdorff sense) divergent curves. Our description of the boundary is slightly simpler construction, which apart from our more specialised notion of rays follows [8]. The term visual boundary is natural in our case, as we use the term visual geometry to mean the study of geodesic rays.

Consider the set of rays emanating from a point $p \in \mathbb{C}\mathbf{H}^2$. Two rays $x(t)$ and $y(t)$ are said to asymptotic, if there is a positive number M such that $d(x(t), y(t)) < M$. On complete Riemannian 3-manifolds the set of rays emanating from a point p is in one-to-one correspondence with the visual sphere $\mathbb{S}^3$ consisting of unit tangent vectors at p. A direct way to see this fact on complex hyperbolic space was described in section 3.1 in the proof of uniqueness of geodesic arcs. This along with the high degree of symmetry of the space $\mathbb{C}\mathbf{H}^2$ lead to a simple description of the visual boundary.

Proposition 4.1. *Two rays on the complex hyperbolic 2-space emanating from the same point are asymptotic if and only if they are equal.*

Proof. Consider the ball model for complex hyperbolic 2-space. Let $x(t)$ and $y(t)$ be rays in $\mathbb{C}\mathbf{H}^2$. By transitivity of isometries of $\mathbb{C}\mathbf{H}^n$, we may assume that $p = 0$, in which case the rays are straight line segments connecting 0 and some point on the bounding sphere $\mathbb{S}^3$. By remarks made in section 3.1, the parametrisations for the rays are given by $x(t) = f(t)\xi$ and $y(t) = g(t)\eta$ for some increasing continuous functions tending to 1 on $\mathbb{R}$ and points ξ and η in $\mathbb{S}^3$. Hence the distance of two points on the rays at time t is given by

$$d(x(t), y(t)) = \frac{1 - f(t)g(t)(\xi, \eta)}{\sqrt{(1 - f(t)^2)(1 - g(t)^2)}}. \tag{4.1}$$

The quantity above approaches infinity, if the limit points are distinct. If it is bounded, it is clear that the two rays must be equal. □

Hence the intuitive idea of the asymptotic boundary corresponds to our definition. The points on sphere $\mathbb{S}^3$ correspond uniquely to classes of geodesic rays. In fact, we may even consider an arbitrary set of isometric embeddings of $\mathbb{R}$ into the space $\mathbb{C}\mathbf{H}^2$ and obtain the same asymptotic boundary set as the quotient under the asymptoticity equivalence relation [9]. We wish, however, to stick to the set of rays emanating from a point, because this way the identification with the visual sphere (the unit tangent space) becomes obvious.

Let (X, d) be a metric space. For $x, y, p \in X$ define the *Gromov product* to be

$$(x|y)_p = \frac{1}{2}\left((d(x,p) + d(y,p) - d(x,y)\right)$$

Extend the Gromov product to the set of rays by setting

$$\left(x(t)|y(t)\right)_p = \lim_{t\to\infty} \left(x(t)|y(t)\right)_p,$$

where we think of $x(t)$ and $y(t)$ as the ray and its value at t. The limit exists for rays on $\mathbb{C}\mathbf{H}^2$ [9].

There is a canonical way of making $\partial_\infty\mathbb{C}\mathbf{H}^2$ into a metric space by defining the metric

$$\rho_p = \lim_{t\longrightarrow\infty} \exp\left(-(x(t)|w(t))_p\right) \tag{4.2}$$

We will call a metric obtained by this construction a *visual metric*.

Our next objective is to compute the visual metric for the ideal boundary $\partial_\infty\mathbb{C}\mathbf{H}^2$ of complex hyperbolic space. We stick to the ball model from now on. Let ξ and η denote points on the unit sphere $\mathbb{S}^3$ corresponding to geodesic rays $x(t)$ and $y(t)$, respectively. A natural choice of base point is 0 for the ease of parametrising the rays. The Gromov product can be evaluated as from the equation

$$\rho_0(\xi, \eta) = \lim_{t\to\infty} \exp(-(x(t)|y(t))_0.$$

After substituting the distance function defined in 3.1, the expression $\exp(-(x(t)|y(t))_0$ takes the form

$$\left(\frac{|1-(x,y)|+\sqrt{|1-(x,y)|^2-(1-|x|^2)(1-|y)^2}}{(1+|z|)(1+|w|)}\right)^{1/2},$$

which at the limit tends to

$$\rho_0(\xi,\eta)=\sqrt{\frac{|1-(\xi,\eta)|}{2}}. \tag{4.3}$$

This generalisation of the chordal metric was defined by Mostow in [11] in the course of proving a rigidity results for a class of Lie groups, which includes groups acting on the complex hyperbolic space. We will use this metric to describe a connection between visual geometry on the space $\mathbb{C}\mathbf{H}^2$ and conformal geometry of the Heisenberg group. The base point will be kept fixed at the origin and omitted from the notation.

Proposition 4.2. *The Cayley transform* $C\colon (\mathbb{H}, d_{\mathbb{H}}) \longrightarrow (\mathbb{S}^3\backslash\{(0,-1)\}, \rho)$ *is a conformal map.*

Proof. We will prove this fact by showing that the metric ρ defines the same infinitesimal structure as the Heisenberg metric. Both metrics are left-invariant, as the visual metric was defined in terms of the complex hyperbolic distance. Hence it suffices to compare distances from a single fixed point.

We will work in Heisenberg coordinates for the punctured sphere $\mathbb{S}^3\backslash\{(0,-1)\}$. Let p be the point $(1,0)$ corresponding to the origin $(0,0,0)\in\mathbb{H}$ and $q=(q_1,q_2)$ is an arbitrary point on the punctured sphere.

Given these choices, the metric ρ has the form

$$\rho(0,q)=\sqrt{\frac{|1-q_2|}{2}}.$$

In coordinates, the coordinate q_2 has the expression

$$q=\frac{1-|z|^2+\mathrm{i}v}{1+|z|^2-\mathrm{i}v},$$

hence ρ has the expression

$$\begin{aligned}\rho(0,(z,v)) =& \sqrt{\left|\frac{|z|^2 - \mathrm{i}v}{1+|z|^2-\mathrm{i}v}\right|}\\ =& \left(\frac{|z|^4+v^2}{(1+|z|^2)^2+v^2}\right)^{1/4}\\ =& \frac{d_{\mathbb{H}}(0,(z,v))}{1+\mathcal{O}(d_{\mathbb{H}}(0,(z,v)))^2}\end{aligned}$$

on the Heisenberg group $\mathbb{H}$. This shows that the two metrics $d_{\mathbb{H}}$ and ρ are the same up to a first order approximation for two points that are close to each other. Hence the limits $\limsup_{r\to 0} a(0,r)$ and $\liminf_{r\to 0} b(0,r)$ for the identity mapping $(\mathbb{H},\rho) \longrightarrow (\mathbb{H},d_C)$ must be the same. This proves our claim. $\square$

Finally, we state the formal conclusion that conformal groups of the two spaces are identified by the Cayley transform, as conformality is preserved in composition.

Corollary 4.1. *The class of conformal mappings on the space* $(\mathbb{S}^3 \backslash \{(1,0)\},\rho)$ *is obtained by conjugating the conformal group of* $\mathbb{H}$ *by the Cayley transform.*

Notice that the visual metric gives us a fourth way to define the same length metric structure, besides the usual CC construction, one using the Heisenberg metric and the collapsing of the hyperbolic metric at the boundary.

5 Conclusions

The Heisenberg group was shown to be a sub-Riemannian manifold with nontrivial metric structure, whose geometry is at each point an extension of the planar geometry. The algebraic study of geodesics was greatly simplified by the use of exponential coordinates. We dealt with the complex hyperbolic space and especially how the Heisenberg group appears as an embedded subgroup. The sub-Riemannian geometry of the Heisenberg group appeared at the boundary of the ambient complex hyperbolic space by collapsing the hyperbolic metric, which suggested the technique of Riemannian approximation. The identification of the Heisenberg group with a compact topological boundary of the complex hyperbolic space allowed the two-way traffic of ideas and algebraic expressions between the sub-Riemannian and complex hyperbolic worlds. The visual viewpoint yielded another way to describe the differences and similarities of (locally) euclidean and sub-Riemannian spaces. We also saw how the language could be applied to asymptotic geometry on the complex hyperbolic space to find a connection to the conformal geometry of the sub-Riemannian Heisenberg group.

While many of the results we presented were not fully proved, an attempt was made to either mention the idea of the proof or cite a source for a complete proof. The Heisenberg group has extensively studied and many different aspects of it have appeared in papers and books. Many texts contain the skeleton of the same facts we have discussed, but we have tried to write out some constructions and proofs in more detail than what could be found in the sources cited.

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