Tensor operations, convex bodies and K-stability

Anton Isopoussu

August 2, 2019

Abstract

We define the notions of a convex combination of a test configuration, and K-stability over a base variety.

1 Introduction

The K-stability of a projective variety with the structure of a projective family over a base scheme is in certain cases conjecturally characterised in terms of two types of simple test configurations. On the one hand one can look at test configurations which are equivariant with respect to the projection to the base, and on the other hand one can pull back test configurations from the base. Partial results are known in the case of toric bundles [2], projective bundles [20], blowups [3, 25, 20] and flag bundles [**?**]. We define the notion of *relative K-stability*, which is a conjectural refinement of K-stability. Given a projective morphism $p\colon Y \to B$ a *relative test configuration* is a projective morphism $\mathscr{Y} \to B \times \mathbb{A}^1$, with a $\mathbb{G}_m$-action inducing a test configuration on each fibre of p.

We introduce and study filtrations of graded coherent sheaves of algebras in Section 2 with the aim of generalising the Witt-Nyström-Székelyhidi theory of filtrations in the study of K-stability [29, 27] to the context of relative K-stability. We show how this relates to Székelyhidi's notion of $\overline{\mathrm{K}}$-stability (see Remark **??**) in Section 3. The motivation for studying filtrations of sheaves is that it allows us to give a unified treatment of several constructions that have appeared in the theory of K-stability, as well as constructions which we believe to be new. Related work was done by Ross and Thomas [21].

In Section 4, we propose an algebraic solution to the problem of interpolating test configurations, which was solved analytically in [23]. This is an application of the constructions defined in Section 2 and Section 3. Our approach works when the test configurations are defined for different polarisations as well. As an application, we prove that the K-unstable locus in $\mathbb{V}(X)$ is open in the Euclidean topology. The behaviour of convex transforms as well as further examples of the interpolation construction are studied in Section 5.

In Section 6, we apply the constructions to give a natural definition of pulling back test configurations from the base scheme B. We also give an overview where test configurations of this type have appeared in the literature. Finally, we discuss natural filtrations of the coordinate algebras of flag bundles from the new point of view in Section 7.

Remark 1 (A note on terminology). Throughout this paper, the word *relative* refers to working over a base scheme, not to be confused with the stability notion used in the extremal YTD correspondence.

Remark 2. As far as we know, apart from Theorem 26 and Proposition 30 (Theorem **??**), our results are new even when working over $\operatorname{Spec}\mathbb{C}$.

2 Filtrations and projective families

By convention, our algebras are $\mathbb{Z}^n_{\geq 0}$-graded. Let B be a scheme over the complex numbers. If $\mathcal{A}$ is a graded sheaf of $\mathcal{O}_B$-algebras, we assume that $\mathcal{A}_0 = \mathcal{O}_B$.

Definition 3 (Admissible filtrations). Let

$$\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k \tag{1}$$

be a sheaf of quasicoherent graded $\mathcal{O}_B$-algebras over a scheme B. Then an *admissible filtration* of $\mathcal{A}$ is a filtration of coherent subsheaves

$$F_\bullet : 0 = F_{-1}\mathcal{A} \subset \mathcal{O}_B = F_0\mathcal{A} \subset F_1\mathcal{A} \subset \cdots \subset \mathcal{A}, \tag{2}$$

such that it is

(i) *multiplicative*, the filtration satisfies the relation $(F_i\mathcal{A})(F_j\mathcal{A}) \subset F_{i+j}\mathcal{A}$,

(ii) *homogeneous*, if U is an open set in B, the homogeneous parts of any section of $F_i\mathcal{A}(U)$ are all in $F_i\mathcal{A}(U)$, and

(iii) *exhaustive*, it satisfies $\bigcup_{i=0}^{\infty} F_i\mathcal{A} = \mathcal{A}$.

Remark 4. The property $F_0\mathcal{A} = \mathcal{O}_B$ can be replaced by saying that a filtration

$$\cdots \subset F_i\mathcal{A} \subset F_{i+1}\mathcal{A} \subset \cdots \tag{3}$$

is *discrete*, meaning that $F_j\mathcal{A} = \mathcal{O}_B$ for some j. Any such filtration can be uniquely reindexed as an admissible filtration.

There is another equivalent convention for defining an admissible filtration by reversing the order of the filtration. Codogni and Dervan described the process of translating between the two points of view in [6] in the nonrelative case. We work with increasing filtration as a matter of convenience while developing the theory.

Definition 5. Let $\mathtt{FAlg}_{\mathcal{O}_B}$ denote the category of pairs $(\mathcal{A}, F_\bullet\mathcal{A})$ such that

(i) $\mathcal{A}$ is a graded coherent $\mathcal{O}_B$-algebra, which is locally finitely generated over $\mathcal{O}_B$ and

(ii) $F_\bullet\mathcal{A}$ is an admissible filtration.

The morphisms are grading and filtration preserving homomorphisms. We refer to the objects admissibly filtered graded $\mathcal{O}_B$-algebras and often simply refer to them by the symbol $F_\bullet\mathcal{A}$.

Definition 6. Let $f\colon \mathcal{A} \to \mathcal{B}$ be an surjection of graded $\mathcal{O}_B$-modules and f_i is the restriction of f to the subsheaf $F_i\mathcal{A}$. We define the *image filtration* $(f_*F)_\bullet\,\mathcal{B}$ by

$$(f_*F)_i\mathcal{B} = \operatorname{im} f_i. \tag{4}$$

Definition 7. Let $g : \mathcal{A} \to \mathcal{B}$ be a morphism of graded filtered O_B-algebras and let $G_\bullet\mathcal{B}$ be a filtration of $\mathcal{B}$. We define the *induced* filtration $(f^*G)_\bullet\,\mathcal{A}$ by

$$(f^*G)_i\,\mathcal{A} = \mathcal{A} \cap G_i\mathcal{B} = \{a \in \mathcal{A} : f(a) \in G_i\mathcal{B}\}. \tag{5}$$

Remark 8. If f is an isomorphism, these two constructions are clearly inverse to one another, that is we have identities

$$f_*f^*G_\bullet\mathcal{A} = G_\bullet\mathcal{A} \tag{6}$$

and

$$f^*f_*F_\bullet\mathcal{A} = F_\bullet\mathcal{A}. \tag{7}$$

Definition 9. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_B$-modules and let $H_i\mathcal{A} \in \mathtt{FAlg}_{\mathcal{O}_B}$. We define the *derived filtration* [5], also denoted by $H_\bullet\mathcal{E}$, by

$$H_i\mathcal{E} = (H_i\mathcal{A})\mathcal{E}. \tag{8}$$

Lemma 10. *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a (grading-preserving) morphism of filtered graded sheaves of $\mathcal{O}_B$-algebras. Then the image filtration and induced filtration, when defined, are admissible filtrations in the sense of Definition 3.*

Proof. We verify the conditions in Definition 3 starting with the image filtration. Fix a filtered algebra $F_\bullet\mathcal{A} \in \mathtt{FAlg}_{\mathcal{O}_B}$. To show (i), let s_i and s_j be sections of $f_*F_i\mathcal{A}$ and $f_*F_j\mathcal{A}$ over $U \subset B$, respectively. Then making U smaller if necessary, we have elements t_i and t_j in $F_i\mathcal{A}(U)$ and $F_j\mathcal{A}(U)$, respectively, such that $f(t_i) = s_i$ and $f(t_j) = s_j$. The section t_it_j is in $F_{i+j}\mathcal{A}(U)$, so $f(t_it_j)$ is in $(f_*F)_{i+j}\mathcal{A}(U)$. Homogeneity and exhaustivity follow easily since f preserves the grading and is a surjective map of sheaves.

The induced case is similar. To check multiplicativity, let $s_i \in g^*G_i\mathcal{B}(U)$ and $s_j \in g^*G_j\mathcal{B}(U)$. Since $G_\bullet\mathcal{B}$ is admissible and g is a homomorphism, we have $g(s_is_j) \in G_{i+j}\mathcal{B}(U)$ and hence $s_is_i \in g^*G_{i+j}(U)$. Homogeneity and exhaustivity are again trivial, since the map g preserves the grading. $\square$

Tensor algebras of filtered modules are naturally endowed with an admissible filtration.

Definition 11 (The tensor algebra of a filtered module)**.** Let

$$F_\bullet\mathcal{E} : 0 = F_0\mathcal{E} \subset F_1\mathcal{E} \subset \cdots \subset F_n\mathcal{E} = \mathcal{E} \tag{9}$$

be a filtered sheaf of $\mathcal{O}_B$-modules. The tensor algebra of $\mathcal{E}$ is naturally a filtered algebra by setting

$$G_pT(\mathcal{E}) = \mathcal{O}_B \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{i_1+\cdots+i_k=p} F_{i_1}\mathcal{E} \otimes \cdots \otimes F_{i_k}\mathcal{E} \tag{10}$$

for $p \in \mathbb{Z}_{>0}$.

Lemma 12. *The filtration defined in Equation* (10) *is admissible.*

Proof. Follows directly from the definitions. □

Definition 13 (Tensor products of filtered algebras)**.** Let $F_\bullet\mathcal{A}$ and $G_\bullet\mathcal{B}$ be filtered sheaves of graded $\mathcal{O}_B$-algebras. Define the tensor product

$$\left(F_\bullet \otimes G_\bullet\right)_p (\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{B}) = \bigoplus_{i+j=p} F_i\mathcal{A} \otimes_{\mathcal{O}_B} G_j\mathcal{B}, \tag{11}$$

which is a filtered $\mathbb{Z}^2$-graded sheaf of coherent $\mathcal{O}_B$-algebras.

Lemma 14. *Tensor products of filtered algebras are commutative and associative.*

Definition 15. The *Veronese subalgebra* $\mathcal{A}^{(d)}$ is defined as the subalgebra

$$\mathcal{A}_{(d)} = \bigoplus_{k=0}^{\infty} \mathcal{A}_{dk}. \tag{12}$$

Similarly, if $\mathcal{C}$ is a $\mathbb{Z}^N_{\geq 0}$-graded sheaf of algebras, define the $a = (a_1, \ldots, a_N)$-diagonal

$$\mathcal{C}_a = \bigoplus_{k=0}^{\infty} \mathcal{C}_{(ka_1,\ldots,ka_n)}. \tag{13}$$

Definition 16 (Diagonal subalgebras)**.** Let $F_\bullet\mathcal{A}$ and $G_\bullet\mathcal{B}$ be filtered sheaves of graded $\mathcal{O}_B$-algebras. For any pair (a, b) of nonnegative integers, we define the (a, b)-*diagonal product* of the two filtered algebras by

$$\left(F_\bullet \otimes_{(a,b)} G_\bullet\right)(\mathcal{A} \otimes \mathcal{B}) = \left(\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{B}\right)_{(a,b)} \cap \left(F_\bullet \otimes G_\bullet\right)_\bullet (\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{B}). \tag{14}$$

We refer to this filtration the (a, b)-*diagonal product* of two filtered algebras. Define weighted diagonal products of any finite collections of filtered sheaves of algebras similarly.

Lemma 17. *The diagonal product is a well-defined operation on* $\mathtt{FAlg}_{\mathcal{O}_B}$.

Proof. This is a straightforward verification. □

Definition 18 (Filtrations generated at degree 1)**.** Let $F_\bullet\mathcal{E}$ be a filtered sheaf of $\mathcal{O}_B$-modules and $\mathcal{A}$ a graded sheaf of $\mathcal{O}_B$-algebras such that $\mathcal{A}_1 = \mathcal{E}$. We say that the algebra $\mathcal{A}$ is generated at degree 1 so that there is a surjective morphism

$$p\colon S(\mathcal{E}) \to \mathcal{A}. \tag{15}$$

Let $F_\bullet S(\mathcal{E})$ be the filtration on $S(\mathcal{E})$ induced by the filtration on $T(\mathcal{E})$ defined in Definition 11. Define the *filtration* $G_\bullet\mathcal{A}$ *of* $\mathcal{A}$ *generated by* $F_\bullet\mathcal{E}$ to be the image filtration $p_*F_\bullet\mathcal{A}$.

Lemma 19. *A filtration generated at degree 1 is admissible.*

Proof. Follows from Lemma 10 and Lemma 12. □

Definition 20. We define the Rees algebra and the associated graded algebra of $F_\bullet\mathcal{A}$ as

(i) $\mathcal{R}ees(F_\bullet\mathcal{A}) = \oplus_{i\geq 0}(F_i\mathcal{A})t^i \subset \mathcal{A}[t]$,

(ii) $gr(F_\bullet\mathcal{A}) = \oplus_{i\geq 0}(F_i\mathcal{A})/(F_{i-1}\mathcal{A})$,

respectively. We say that a filtration $F_\bullet\mathcal{A}$ is finitely generated if $\mathrm{Rees}(F_\bullet\mathcal{A})$ is locally finitely generated as an $\mathcal{O}_B$-algebra. Note that both objects are bigraded. We refer to the two gradings by the $\mathcal{A}$-grading and the t-grading.

Lemma 21. *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism of graded sheaves of O_B-algebras. The tensor product preserves finite generation of admissible filtrations. If we assume the homomorphism f is surjective, the same is true for the image filtration. Similarly, if the homomorphism f is injective, the induced filtration is finitely generated.*

Proof. This can be easily seen by relating the Rees algebras. Let $F_\bullet$ and $G_\bullet$ be filtrations for $\mathcal{A}$ and $\mathcal{B}$, respectively, and $f\colon \mathcal{A} \to \mathcal{B}$ is a map preserving the grading. Then we have natural morphisms

$$\mathcal{R}ees(F_\bullet\mathcal{A}) \to \mathcal{R}ees(f_*F_\bullet\mathcal{B}) \tag{16}$$

and

$$\mathcal{R}ees(f^*G_\bullet\mathcal{A}) \to \mathcal{R}ees(G_\bullet\mathcal{B}) \tag{17}$$

which preserve the grading. The claims for pushforwards and pullbacks then follow easily. Note that we must assume that f is a surjection in the pushforward case. In the tensor product case we have a natural isomorphism

$$\mathcal{R}ees\left(F_\bullet\mathcal{A} \otimes_{\mathcal{O}_B} G_\bullet\mathcal{B}\right) \cong \mathcal{R}ees(F_\bullet\mathcal{A}) \otimes_{\mathbb{C}[t]} \mathcal{R}ees(G_\bullet\mathcal{B}) \subset (A \otimes B)\,[t] \tag{18}$$

which immediately implies the claim. □

Remark 22 (Filtrations of coordinate rings)*.* Let (B, L) be a projective scheme and denote $R = \bigoplus_{k=1}^{\infty} H^0(B, L^k)$. Definition 3 contains the special case of admissible filtrations as defined [27] in of R by taking the base to be a point.

3 Relative K-stability

In this section we define relative test configurations and describe their relationship to admissible filtrations discussed in Section 2.

Fix a projective scheme B of dimension b with an ample line bundle L and a locally finitely generated graded sheaf of $\mathcal{O}_B$-algebras $\mathcal{A}$. Denote the relative projectivisation of $\mathcal{A}$ by $Y = \mathcal{P}roj_B(\mathcal{A})$ with the projection $p\colon Y \to B$. We assume that $\mathcal{A}$ is locally finitely generated at degree 1, which means that there exists a surjective homomorphism

$$S(\mathcal{A}_1) \to \mathcal{A} \tag{19}$$

and hence an embedding

$$\mathcal{P}roj_B\,\mathcal{A} \to \mathbb{P}\mathcal{A}_1. \tag{20}$$

Definition 23. Define the *graded algebra of sections* of L by

$$R_L = \bigoplus_{k=0}^{\infty} H^0(B, L^k) \tag{21}$$

and the associated graded sheaf of algebras by

$$\mathcal{R}_L = \bigoplus_{k=0}^{\infty} L^k. \tag{22}$$

Proposition 24. *The Rees algebra of a graded sheaf of coherent $\mathcal{O}_X$-algebras*

$$\mathcal{R}ees(F_\bullet\mathcal{A}) = \bigoplus_{k=0}^{\infty} F_k\mathcal{A}t^k \tag{23}$$

is a flat sheaf of graded $\mathcal{O}_{\mathbb{A}^1}$-algebras.

Proof. The claim is local on B. The Rees algebra of a $k[t]$-module is torsion free as a $k[t]$-algebra. A well known flatness criterion states that a module over a principal ideal domain is flat if and only if it is torsion free [9, Section 6.3]. □

We say that $\mathcal{A}$ is *ample* if the $\mathcal{O}(d)$-line bundle on Y defines an embedding for some positive integer d. If this is true for $d = 1$, $\mathcal{A}$ is *very ample.*

Definition 25. Let Y be a scheme, $p\colon Y \to B$ a projective morphism and $\mathcal{L}$ a p-ample line bundle. A *relative test configuration*, or *p-test configuration* $(\mathscr{Y}, \mathscr{L}, \rho)$ for the pair $(Y, \mathcal{L})$ is defined by

- a flat morphism $f\colon \mathscr{Y} \to \mathbb{A}^1$ which factors through $B \times \mathbb{A}^1$, along with an isomorphism $\varphi_t : f^{-1}\{1\} \cong Y$,
- an f-ample line bundle $\mathscr{L}$ on $\mathscr{Y}$ such that $\mathscr{L}_t$ such that the isomorphism over the fibre $f^{-1}\{1\}$ identifies the line bundles $\mathscr{L}_1$ and $\mathcal{L}$.

- an algebraic action $\rho : \mathbb{G}_m \times \mathscr{Y} \to \mathscr{Y}$ which makes the projection to $B \times \mathbb{A}^1$ equivariant with respect to the trivial action on B and the standard action on $\mathbb{A}^1$, together with a $\mathscr{L}$-linearisation action on $\mathscr{Y}$ that covers the usual action on $\mathbb{A}^1$.

The integer r is called the *exponent* of the p-test configuration. The fibre $f^{-1}\{0\}$ is called the central fibre. If $\mathscr{L}$ is ample, a p-test configuration is a test configuration in the sense of Definition **??**, in which case we say that $\mathscr{Y}$ is an ample p-test configuration.

Theorem 26. *A finitely generated admissible filtration $F_\bullet\mathcal{A}$ determines a p-test configuration*

$$\left(\mathcal{P}roj_{B\times\mathbb{A}^1}\, \mathcal{R}ees\, F_\bullet\mathcal{A}, \mathcal{O}(1)\right) \tag{24}$$

with its natural $\mathbb{G}_m$-action. Conversely, a p-relative test configuration $(\mathscr{Y}', \mathscr{L})$ of $\mathcal{P}roj_B\, \mathcal{A}$ determines a finitely generated admissible filtration $G_\bullet\mathcal{A}$.

Proof. Let the group $\mathbb{G}_m$ act with its natural action on the line $\mathbb{A}^1$ and extend it trivially to the product $B \times \mathbb{A}^1$. There is a natural linearisation of this action on the sheaf $\mathcal{R}ees\, F_\bullet\mathcal{A}$ with the following local description. Let U be an open set in B such that the projection $p\big|_U$ corresponds to a graded A_0-algebra A, where A_0 is the coordinate ring of B over U. The filtration $F_\bullet\mathcal{A}$ restricts to an admissible filtration $F_\bullet A$. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Rees}\, F_\bullet A \arrow[r, "t\mapsto s^{-1}t"] & \mathrm{Rees}\, F_\bullet A[s^{\pm 1}] \\ A_0[t] \arrow[u] \arrow[r, "t\mapsto s^{-1}t"] & A_0[t, s^{\pm 1}] \arrow[u] \end{tikzcd}$$

with obvious notation. This defines a $\mathbb{G}_m$-linearisation on $\mathcal{A}$ over U compatible with the grading. The morphisms p_U glue as U ranges over an open cover of B to determine a $\mathbb{G}_m$-scheme $(\mathcal{P}roj_{B\times\mathbb{A}^1}\, \mathcal{R}ees\, F_\bullet\mathcal{A}, \mathcal{O}(1))$ with an equivariant projection down to $B \times \mathbb{A}^1$. The projection to $\mathbb{A}^1$ is flat by Proposition 24 and the central fibre is isomorphic to

$$\mathcal{P}roj_B\, gr(F_\bullet\mathcal{A}) \tag{25}$$

with a $\mathbb{G}_m$-action defined by the t-grading.

Given a p-test configuration $(\mathscr{Y}, \mathscr{L})$, we produce an admissible filtration as follows. By replacing $\mathscr{L}$ with a power if necessary, we may assume that we have an embedding

$$\iota\colon \mathscr{Y} \longrightarrow \mathbb{P}g_*\mathscr{L}, \tag{26}$$

where g is the projection $\mathscr{Y} \to B \times \mathbb{A}^1$. Using the identification $(\mathscr{Y}_1, \mathscr{L}\big|_{B\times\{1\}}) \cong (Y, \mathcal{L})$ we obtain a natural map

$$h\colon \mathcal{A} \longrightarrow \bigoplus_{k=0}^{\infty} g_* \left(\mathscr{L}\big|_{B\times\{1\}}\right)^{\otimes k}, \tag{27}$$

which we may take to be an isomorphism by [24, Lemma 29.14.4].

For any sufficiently small affine neighborhood $U \cong \operatorname{Spec} A_0 \subset B$, we have a diagram

$$\begin{tikzcd} g^{-1}U \cong \operatorname{Proj}_{\operatorname{Spec} A_0} S \arrow[r, "\iota"] \arrow[dr, "g|_U"'] & \mathbb{P}_{\operatorname{Spec} A_0} S_1 \arrow[d] \\ & \operatorname{Spec} A_0 \end{tikzcd}$$

where S is a graded A_0-algebra. Since the projection g is equivariant for the trivial action on U, the linearisation of the $\mathbb{G}$-action determines a representation on A_1. This determines a splitting $A_1 = \bigoplus_{i=1}^r W_i$ by weight. We obtain a presheaf of filtered $\mathcal{O}_B$-modules as U ranges over sufficiently small affine open sets of B. The associated sheaf generates an admissible filtration $G_\bullet \mathcal{A}$ of $\mathcal{A}$ by Lemma 19. $\square$

Remark 27. If $B = \operatorname{Spec} \mathbb{C}$, this theorem was proved by [27].

If $X = Y = B$, L is an ample line bundle on X and p is the identity morphism, this theorem reduces to the blowing up formalism due to Mumford [16], Ross and Thomas [21] and Odaka [17]. Up to passing to a Veronese subalgebra, any finitely generated admissible filtration of the algebra $\mathcal{R}_L$ can be obtained from a filtration

$$\mathcal{I}_1 \subset \cdots \subset \mathcal{I}_N \subset \mathcal{O}_X. \tag{28}$$

See Remark 34 for an outline of this construction.

Given an admissible filtration $F_i\mathcal{A}$ we define the associated Hilbert, weight and trace squared functions by

$$h(k) = \sum_{i=1}^{\infty} \chi\left(B, \frac{F_i\mathcal{A}_k}{F_{i-1}\mathcal{A}_k}\right)$$
$$w(k) = \sum_{i=1}^{\infty} -i\chi\left(B, \frac{F_i\mathcal{A}_k}{F_{i-1}\mathcal{A}_k}\right)$$

and

$$d(k) = \sum_{i=1}^{\infty} i^2\chi\left(B, \frac{F_i\mathcal{A}_k}{F_{i-1}\mathcal{A}_k}\right),$$

respectively. If the p-test configuration given by Theorem 26 is ample, the functions $h(k)$, $w(k)$ and $d(k)$ are equal to the functions defined in Lemma **??**. In this case the Donaldson-Futaki invariant is defined normally by Equation (**??**).

Definition 28 (Relative K-stability)**.** Let $Test_B(Y, L)$ be the set of p-test configurations of (Y, L). We define *K-stability relative to* p in the same way we defined K-stability in Definition **??** but by restricting the set of test configurations to ones which lie in $Test_B(Y, L)$.

Definition 29. Consider the equivalence relation on the set of p-test configurations generated by the following three relations.

(i) Identify a p-test configuration $\mathscr{Y}$ with any test configuration with which it is $\mathbb{G}_m$-equivariantly isomorphic.

(ii) Identify any rescaling of the $\mathbb{G}_m$-action on $(\mathscr{Y}, \mathscr{L})$ (pullback by a cover of $\mathbb{A}^1$, cf. Remark **??**).

(iii) Identify any pair $(\mathscr{Y}, \mathscr{L})$ and $(\mathscr{Y}, \mathscr{L}^d)$ of p-test configurations for all $d > 1$.

Following Odaka [18] we call equivalence classes under the above identifications *p-test classes* for test configurations. Test configurations up to the first two relations are called *p-test degenerations*. Note that we will use the same terminology for arbitrary filtrations later, see Definition 36.

Proposition 30 (Theorem **??**)**.** *The two constructions in Theorem 26 induce a 1-1 correspondence between finitely generated filtrations of $\mathcal{A}$ up to isomorphism and Veronese subalgebras, and p-test classes of $(Y, \mathcal{L})$.*

Proof. It suffices to show that the two constructions are inverses to one another up to the stated identifications.

An automorphism φ of a filtered algebra $F_\bullet\mathcal{A}$ induces an automorphism of the Rees algebra, and hence of its projectivisation. Conversely, any equivariant isomorphism which preserves linearisations clearly produces an automorphism of the filtered algebra.

Similarly, the admissibility criterion uniquely fixes the scale of the action, while the final identification corresponds to identifying Veronese subalgebras of $F_\bullet\mathcal{A}$. This completes the proof. □

We extend the notion of ampleness to admissible filtrations through ampleness of their *finitely generated approximations.*

Definition 31 (Ampleness for filtrations)**.** Let $F_\bullet\mathcal{A}$ be the filtered algebra and define the filtrations $F_\bullet^{(k)}\mathcal{A}$ for all $k \in \mathbb{N}$ to be the filtrations of $\mathcal{A}_{(k)}$ generated by the filtration $F_\bullet\mathcal{A}_k$. We say that an element of $\mathtt{FAlg}_{\mathcal{O}_B}$ is *ample* if the sequence of filtrations $F_\bullet^{(k)}\mathcal{A}$ determine p-ample test configurations for all $k \in \mathbb{N}$.

Definition 32. For any line bundle A on B, define the *twisted polarisation*

$$\mathcal{L}(A) = \mathcal{L} \otimes p^* A. \tag{29}$$

We abuse notation by denoting the twisted polarisation on any test configuration of Y similarly.

Lemma 33. *Let $(\mathscr{Y}, \mathscr{L})$ be a p-test configuration for (Y, L) and let L be an ample line bundle on B. Then $(\mathscr{Y}, \mathscr{L}(L^m))$ is ample for $m \gg 0$.*

Proof. It suffices to check ampleness over the central fibre $B \times \{0\}$, over which the line bundle $\mathscr{L}(L^m)$ restricts to $\mathcal{F}(L^m)$ for some relatively ample line bundle $\mathcal{F}$ by construction. This is ample by [11, Proposition II.7.10]. $\square$

We close the section on a brief discussion of slope stability which provides a case where amplitude has been studied in detail in Ross and Thomas [20].

Remark 34 (Slope stability). Let $\iota : B' \subset B$ is a subscheme. We define a filtration of $\mathcal{R}$ by vanishing orders along B'. Denote the ideal sheaf of B' by $\mathscr{I}_{B'}$ and consider the filtration

$$G_\bullet L^a \colon \mathscr{I}^b L^a \subset \mathscr{I}^{b-1} L^a \subset \cdots \mathscr{I} L^a \subset L^a \tag{30}$$

for any pair of natural numbers a and b. Assume from now on that a and b are coprime. The tensor algebra generated by $G_\bullet L^a$ (cf. Definition 18) is admissibly filtered by Lemma 12.

For example, if $a = b = 1$ we write

$$\begin{aligned}
\mathcal{O}_B &\subset \mathscr{I} L \oplus \mathscr{I}^2 L^2 \oplus \mathscr{I}^3 L^3 \oplus \mathscr{I}^4 L^4 \oplus \cdots \\
&\subset L \oplus \mathscr{I} L^2 \oplus \mathscr{I}^2 L^3 \oplus \mathscr{I}^3 L^4 \oplus \cdots \\
&\subset L \oplus L^2 \oplus \mathscr{I} L^3 \oplus \mathscr{I}^2 L^4 \oplus \cdots \\
&\qquad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
&\subset \mathcal{R} = L \oplus L^2 \oplus L^3 \oplus L^4 \oplus \cdots \qquad\qquad .
\end{aligned}$$

It is easy to pick out the filtration from the increasing sequence of upper triangular subsets starting from the top left corner starting with

$$\mathcal{O}_B \subset (O_B \oplus \mathscr{I} L) \subset \left(\mathcal{O}_B \oplus L \oplus \mathscr{I}^2 L^2\right) \subset \cdots . \tag{31}$$

We denote the associated p-test configuration by $\mathscr{X}_c$ for $c = \frac{a}{b}$. If we assume that $c \leq \mathrm{Sesh}(B', L)$, where

$$\mathrm{Sesh}(B', L) = \sup\left\{c \in \mathbb{Q}_{>0} \colon L^r \otimes \mathscr{I}_{B'}^{cr} \text{ is globally generated for } r \gg 0\right\}, \tag{32}$$

then the p-test configuration $\mathscr{X}_c$ is ample (up to an equivariant contraction in the case $c = \mathrm{Sesh}(B', L)$). This fact is due to Ross and Thomas, who also found a beautiful formula for the Donaldson-Futaki invariant in this case in terms of the *slope* of the triple (B', L, c)[1] [20].

More complicated filtrations of the structure sheaf also yield admissible filtrations in a similar manner. Conversely, let $F_\bullet \mathcal{R}_L$ be an admissible filtration which is generated in degree 1. Let N be the smallest integer such that $F_N L = L$. For any $1 \leq i \leq N$, we can define the ideal sheaf $\mathscr{I}_i \subset \mathcal{O}_X$ to be the ideal sheaf locally generated by sections of the subsheaf $F_i L$. We obtain a filtration

$$0 \subset \mathscr{I}_1 \subset \cdots \subset \mathscr{I}_N \subset \mathcal{O}_X. \tag{33}$$

An alternative construction of the ideal sheaves $\mathscr{I}_i$, starting with an arbitrary test configuration, can be found in Odaka [17, Proposition 3.10] or Ross and Thomas [21].

[1]Proposition **??** is proved using this formula.

4 Convex combinations of test configurations

The aim of this section is to define a convex structure on equivalence classes of test configurations. The idea is very simple and is based on Segre products of filtered coordinate algebras. Consider the following example.

Example 35 (A description of the convex combination of test configurations when the base B is a point)**.** Let V and W be complex vector spaces and let X be a projective variety with two embeddings $\iota_1 \colon X \subset \mathbb{P}(V)$ and $\iota_2 \colon X \subset \mathbb{P}(W)$. Fix two 1-parameter subgroups of $\mathrm{SL}(V)$ and $\mathrm{SL}(W)$, which determine actions

$$\alpha : \mathbb{P}(V) \times \mathbb{G}_m \to \mathbb{P}(V)$$

and

$$\beta : \mathbb{P}(W) \times \mathbb{G}_m \to \mathbb{P}(W),$$

respectively, and fix two positive integers a and b. Then we have closed immersions

$$X \xrightarrow{\Delta} X \times X \to \mathbb{P}(S^a V \otimes S^b W) \tag{34}$$

and an associated family

$$X \times \mathbb{G}_m \xrightarrow{\Delta} X \times X \times \mathbb{G}_m \subset \mathbb{P}(S^a V \otimes S^b W) \times \mathbb{G}_m. \tag{35}$$

Here the $\mathbb{G}_m$-action on $S^a V \otimes S^b W$ is induced from $\alpha \colon t \mapsto \alpha_t$ and $\beta \colon t \mapsto \beta_t$ by setting

$$(\alpha, \beta)_t (v_1 \otimes \cdots \otimes v_a \otimes w_1 \otimes \cdots \otimes w_b) = (\alpha_t v_1 \otimes \cdots \otimes \alpha_t v_a \otimes \beta_t w_1 \otimes \cdots \otimes \beta_t w_b). \tag{36}$$

We define the *weighted product test configuration* to be the Zariski closure of the image of the diagonal in Equation (35). This is clearly a test configuration for $(X, L_1^a \otimes L_2^b)$, where L_1 and L_2 are the two restrictions of the hyperplane bundle under the embeddings ι_1 and ι_2, respectively.

We write the resulting test configuration additively as

$$a[\alpha] + b[\beta], \tag{37}$$

where the brackets denote taking the product test configuration associated to the $\mathbb{G}_m$-action under the respective embeddings of X into projective space. The test class determined by Equation (35) (cf. Definition 29 and Remark **??**) can be written as

$$(1 - t)[\alpha] + t[\beta], \tag{38}$$

where the parameter t is taken to be $\frac{b}{a+b}$.

From now on, we identify the set of p-test configurations of Y with the set of admissibly filtered algebras $F_\bullet \mathcal{A}$ which satisfy $\mathcal{P}roj_B\,\mathcal{A} \cong Y$ and whose filtration $F_\bullet \mathcal{A}$ is finitely generated by Theorem 26. This justifies the following definition, modelled after Odaka [18].

Definition 36 (Test degenerations and test classes)**.** Let $p\colon Y \to B$ be a projective morphism of normal schemes. Define the set of *p-test degenerations of* Y to be the set $Test_B(Y)$ of admissibly filtered elements $F_\bullet \mathcal{A} \in \mathtt{FAlg}_{\mathcal{O}_B}$ such that $\mathcal{P}roj_B\,\mathcal{A} \cong Y$ considered up to isomorphisms.

Also define the set $\overline{Test_p(Y)}$ of p-test classes by additionally identifying Veronese subalgebras in $Test_p(Y)$. We have a natural map

$$Test_p(Y) \longrightarrow \overline{Test_p(Y)}. \tag{39}$$

If we wish to fix a relatively ample line bundle $\mathcal{L}$ on Y (respectively, a ray of relatively ample line bundles), we write $Test_p(Y,\mathcal{L})$ (resp. $\overline{Test_p(Y,\mathcal{L})}$) for elements of $Test_p(Y)$ (resp. $\overline{Test_p(Y)}$) which define a test degenerations (resp. test classes) for $(Y,\mathcal{L})$.

We denote $Test_{\operatorname{Spec}\mathbb{C}}(B) = Test(B)$.

We now state and prove Theorem **??**. Let $I_{\mathbb{Q}}$ denote the unit interval $[0,1]\cap\mathbb{Q}$ and let Δ_{N-1} be the $N-1$ dimensional simplex in $\mathbb{Q}^N$ defined by $t_1 + \ldots + t_N = 1$ and $t_i \geq 0$ for $i = 1,\ldots,N$.

Theorem 37 (Convex combinations of test configurations)**.** *For any $N \in \mathbb{Z}_{\geq 2}$, there exists a map*

$$\operatorname{Conv}_N\colon\ Test_p(Y)^N \times \Delta_{N-1} \longrightarrow \overline{Test_p(Y)} \tag{40}$$

satisfying

(i) $\operatorname{Conv}_N(\tau, e_i) = \tau_i$*, where e_i is the ith unit vector and $\tau = (\tau_1,\ldots,\tau_N)$ are p-test configurations of $(Y,\mathcal{L}_i)$,*

(ii) $\operatorname{Conv}_N(\tau,t)$ *is an element of $\overline{Test_p(Y,\mathcal{L}_t)}$, where $\mathcal{L}_t$ is the line bundle $\sum_{i=1}^N t_i\mathcal{L}_i$, and*

(iii) if we take $B = \operatorname{Spec}\mathbb{C}$ and assume that τ_i are finitely generated, the Donaldson-Futaki invariant of $\operatorname{Conv}_N(\tau,t)$ *is continuous in the second variable.*

Theorem 38 (Theorem **??**)**.** *The K-unstable locus in $\mathbb{V}(X)$ (cf. Equation (**??**)) is open in the Euclidean topology.*

Proof. Fix a basis $L_1,\ldots,L_N$ of the Picard group of X and let L be a K-unstable polarisation. Fix a test configuration $\mathscr{X}$ for (X,L) with negative Donaldson-Futaki invariant. Let t be a point in $I_{\mathbb{Q}}^N$, $s = 1 - \sum_{i=1}^N t_i$ and let U be a neighbourhood of 0 in $I_{\mathbb{Q}}^N$ such that $(1-s)L + \sum_{i=1}^N t_iL_i$ is ample for all $t \in \mathscr{U}$.

For any $t \in U$, define the test class $[\mathscr{X}_t] = (1-s)[\mathscr{X}] + \sum_{i=1}^N t_i[\mathscr{X}_i]$, where $\mathscr{X}_i$ are trivial test configurations for (X,L_i). By Theorem 37, there is an open neighbourhood $V \subset U$ of 0 such that $\mathrm{DF}(\mathscr{X}_t)$ is negative for all $t \in V$. The set V determines an open neighbourhood of L in $\operatorname{Amp}(X)$ of K-unstable polarisations. Since L was an arbitrary K-unstable polarisation, this completes the proof. ∎

Remark 39. It makes sense to extend the definition of the Donaldson-Futaki invariant of a weighted product $(1-t)\tau_1 + t\tau_2$ for irrational values of t by continuity.

For simplicity of exposition we restrict to the case a pairwise convex combination. The proof of the general case of Theorem 37 follows the same argument with minor adjustments which are outlined in Remark 45 and Remark 46.

Recall first a basic algebraic fact.

Lemma 40. *Let $f\colon S \to T$ be homomorphism of commutative rings and let A and B be T-algebras. Let A_S and B_S be the S-algebras determined by the map f. Then there is a natural surjective homomorphism*

$$g\colon A_S \otimes_S B_S \to A \otimes_T B. \tag{41}$$

Proof. The tensor product $A_S \otimes_S B_S$ is a quotient of $A \otimes_{\mathbb{Z}} B$ by the ideal generated by elements $f(s)a \otimes b - a \otimes f(s)b$ for $s \in S$, $a \in A$ and $b \in B$. This ideal is contained in the ideal of $A \otimes_T B$ in $A \otimes_{\mathbb{Z}} B$, hence identifying both algebras in Equation (41) as quotients of $A \otimes_{\mathbb{Z}} B$ yields the claim. □

Lemma 41 ([13, Example 1.2.22]). *Let L_1 and L_2 be ample line bundles on a projective scheme X. Then the natural map*

$$H^0(X, L_1^a) \otimes_{\mathbb{C}} H^0(X, L_2^b) \longrightarrow H^0(X, L_1^a \otimes L_2^b) \tag{42}$$

is surjective for $a, b \gg 0$.

Corollary 42. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be p-ample line bundles on Y. Then the natural map*

$$p_*\mathcal{L}_1^a \otimes_{\mathcal{O}_B} p_*\mathcal{L}_2^b \longrightarrow p_*\left(\mathcal{L}_1^a \otimes_{\mathcal{O}_Y} \mathcal{L}_2^b\right) \tag{43}$$

is surjective for $a, b \gg 0$.

Proof. By [11, Corollary 12.9] we may assume that the pushforwards $p_*\mathcal{L}_1^a$, $p_*\mathcal{L}_2^b$ and $p_*(\mathcal{L}_1^a \otimes \mathcal{L}_2^b)$ are vector bundles on B. It suffices to check that the map in Equation (43) is surjective on fibres, which follows from 41. □

Let (a, b) be a pair of nonnegative integers and $F_\bullet\mathcal{A}$ and $G_\bullet\mathcal{B}$ two elements of $\mathtt{FAlg}_{\mathcal{O}_B}$ with chosen isomorphisms

$$\mathcal{P}roj_B\,\mathcal{A} \cong Y \;\text{ and }\; \mathcal{P}roj_B\,\mathcal{B} \cong Y. \tag{44}$$

Write $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{B}}$ for the graded algebras associated to the two Serre line bundles. We have natural morphisms

$$\mathcal{A} \to p_*\mathcal{R}_{\mathcal{A}} \;\text{ and }\; \mathcal{B} \to p_*\mathcal{R}_{\mathcal{B}}. \tag{45}$$

By [24, Lemma 29.14.4], there exists a $k_0 > 0$ such that the maps in Equation (45) are isomorphisms in degrees larger than k_0. Therefore the map

$$\varphi : \mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{B} \longrightarrow p_*\mathcal{R}_{\mathcal{A}} \otimes_{\mathcal{O}_B} p_*\mathcal{R}_{\mathcal{B}} \tag{46}$$

is an isomorphism in degrees larger than k_0. Using the isomorphisms in Equation (44) and Corollary 42, we obtain a surjective morphism

$$\varphi : \mathcal{A}_{(a)} \otimes_{\mathcal{O}_B} \mathcal{B}_{(b)} \longrightarrow p_* \left((\mathcal{R}_{\mathcal{A}})_{(a)} \otimes_{\mathcal{O}_Y} (\mathcal{R}_{\mathcal{B}})_{(b)} \right) \tag{47}$$

for $a, b > k_0$.

We will from now on use a mix of additive and multiplicative notation for both test degenerations and line bundles.

Definition 43. For any nonnegative integers a and b we define the *weighted product* of two test degenerations

$$a[F_\bullet \mathcal{A}] + b[G_\bullet \mathcal{B}] \tag{48}$$

to be given by the filtration

$$\varphi_*(F_\bullet \otimes_{(ma,mb)} G_\bullet)(\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{B}), \tag{49}$$

where φ_* denotes taking the image filtration defined in Definition 6 and m is chosen to be the smallest integer so that the statement of Corollary 42 and surjectivity of Equation (45) hold.

Theorem 44. *If τ_1 and τ_2 are p-test degenerations for the relatively ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, the diagonal product determines a p-test configuration for each polarisation on the line segment between $\mathcal{L}_1$ and $\mathcal{L}_2$ in the cone $\mathbb{V}(Y)$ of polarisations (cf. Equation* (**??**)*).*

Proof. This follows from Lemma 16 and the fact that we have

$$\left(\mathcal{P}roj_B \bigoplus_{k=0}^{\infty} p_* \left(\mathcal{L}_1^{ak} \otimes \mathcal{L}_2^{bk} \right), \mathcal{O}(1) \right) \cong \left(Y, \mathcal{L}_1^{ak} \otimes \mathcal{L}_2^{bk} \right). \tag{50}$$

□

Remark 45 (Diagonals in finite products of algebras). Diagonal products make sense for products of three or more elements of $\mathtt{FAlg}_{\mathcal{O}_B}$. First of all, Lemma 41 and Corollary 42 generalise to finite products of line bundles of the form $L_1^{a_1} \otimes \cdots \otimes L_N^{a_N}$ by an easy induction. This avoids the difficulty of having to make a choice of integer m in the construction of the convex combinations of test configurations several times.

In particular, if $F_\bullet\mathcal{A}$, $G_\bullet\mathcal{B}$ and $H_\bullet\mathcal{C}$ are in $\mathtt{FAlg}_{\mathcal{O}_B}$, the (a, b, c) diagonal can be written as a product pairwise diagonals as

$$\begin{aligned} F_\bullet \otimes_{(a,b)} G_\bullet \otimes_{(1,c)} H_\bullet &= F_\bullet \otimes_{(a,1)} \otimes G_\bullet \otimes_{(b,c)} H_\bullet \\ &= F_\bullet \otimes_{(a,1)} \otimes H_\bullet \otimes_{(c,b)} G_\bullet, \end{aligned} \tag{51}$$

where we omit writing the algebra $\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{B} \otimes_{\mathcal{O}_B} \mathcal{C}$. The products are clearly associative so we have omitted the parentheses. Verifying Equation (51) only needs to be done at the level of the diagonal subalgebras, since the filtration on diagonal is simply the restriction of the tensor product filtration. The two identities generate the natural associativity and commutativity properties of the pairwise diagonal product in $\mathtt{FAlg}_{\mathcal{O}_B}$. The same relations descend to the weighted products in $Test_B(Y)$.

Remark 46. There are several potentially confusing aspects about the previous definitions. First, it makes sense to reparametrise the *test class* represented by $a\tau + b\tau$ by rational numbers in the interval $I_{\mathbb{Q}}$. However, convex combinations are *not* well defined for test classes since the diagonal product is clearly not invariant under replacing one of the filtered algebras by a Veronese subalgebra

Second, in order to define the filtration associated to the weighted product, we needed to assume that a and b were sufficiently large in order to make the multiplication maps in Lemma 41 and Corollary 42 surjective. This can be circumvented by replacing both underlying line bundles by a common power at the outset.

Third, while our construction gives no way of choosing a unique convex combination in $Test_B(Y)$, we see no need to do this. We are ultimately interested in test classes. By Remark 45, a convex combination of multiple elements of $Test_B(Y)$ can be done simultaneously and there is no need to iterate a pairwise construction. For test degenerations $\tau = ([F^1\mathcal{A}^1], \ldots, [F^N\mathcal{A}^N])$ and rational numbers

$$t = (t_1, \ldots, t_N) \in \Delta_{N-1} \subset I_{\mathbb{Q}}^N \tag{52}$$

we define $\mathrm{Conv}_N(\tau, t)$ to be the test class of the $(mt_1, \ldots, mt_N)$-diagonal in the filtered algebra

$$\bigotimes_{i=1}^{N} F_\bullet \mathcal{A}^i, \tag{53}$$

where m is a sufficiently large and divisible integer.

We summarise the contents of Remark 45 and Remark 46 in the following proposition.

Proposition 47. *Given N elements of $Test_B(Y)$, there is uniquely defined map from I^N to the set of test classes of Y relative to p. This map is naturally fibred over a subset of the set of rays of p-ample line bundles on Y.*

Before proving property (iii) of Theorem 37 we state the following lemmas. Donaldson reduced the calculation of the total weight to an nonequivariant calculation. See also [22, Section 2.8.1] for a clear exposition.

Lemma 48. *Let X_0 be a projective $\mathbb{G}_m$-scheme over the complex numbers with an ample $\mathbb{G}_m$-linearised line bundle L. Then there exists a polarised scheme $(\mathcal{Y}, \mathcal{H}_L)$ such that the the weight polynomial is given by*

$$\mathrm{tr}\, H^0(X_0, L^k) = \chi(\mathcal{Y}, \mathcal{H}_L^k) - \chi(X_0, L^k). \tag{54}$$

Dervan proved the following generalisation of Donaldson's formula.

Lemma 49 ([7, Lemma 2.30 (iv)])**.** *Keep the notation of Lemma 48 and let A be a $\mathbb{G}_m$-linearised line bundle on X_0. The total weight of the $\mathbb{G}_m$-representation on the vector space $H^0(X_0, L^k \otimes A)$ is given by*

$$\mathrm{tr}\, H^0(X_0, L^k \otimes A) = \mathrm{tr}\, H^0(X_0, L^k) - \int_{\mathcal{Y}} \frac{c_1(\mathcal{H}_L)^n \cdot c_1(\mathcal{H}_A)}{n!} k^n + O(k^{n-1}), \tag{55}$$

for some line bundle $\mathcal{H}_A$ on $\mathcal{Y}$.

Corollary 50. *Keep the notation of Lemma 48 and let L_i be ample $\mathbb{G}_m$-linearised line bundles on X_0 for $1 \leq i \leq N$. We have an identity*

$$\operatorname{tr} H^0(X_0, \bigotimes_{i=1}^{N} L_i^{a_i k}) = C_0(a_1, \ldots, a_N)k^{n+1} + C_1(a_1, \ldots, a_N)k^n + O(k^{n-1}). \tag{56}$$

where $C_0(a_1, \ldots a_N)$ and $C_1(a_1, \ldots, a_N)$ are polynomials in $a_1, \ldots, a_N$.

Proof. Apply Lemma 49 and Lemma 48 to

$$L = L_j^k \text{ and } A = \bigotimes_{i=1, i\neq j}^{N} L_j^{a_i k} \tag{57}$$

for $j = 1, \ldots, N$. □

Claim 51. *Property (iii) of Theorem 37 holds.*

Proof. We show that the Donaldson-Futaki invariant is a continuous rational function in t for $t \in \Delta_{N-1}$.

By the Riemann-Roch formula, there exist polynomials c_0 and c_1 in a_i such that

$$h^0(X, \bigotimes L_i^{a_i k}) = c_0 k^n + c_1 k^{n-1} + O(k^{n-2}). \tag{58}$$

In particular, there exist positive numbers $c_{0,i}$ such that

$$c_0 = \sum_{i=1}^{N} c_{0,i} a_i^n + O(a_1^{n-1}, \ldots, a_N^{n-1}), \tag{59}$$

since L_i are all ample.

By Corollary 50, the weight function is similarly a polynomial in the a_i. We conclude that the function

$$t \mapsto \mathrm{DF}\left(t_1\tau_1 + \cdots + t_{N-1}\tau_{N-1} + (1 - \sum_{i=1}^{N-1} t_i)\tau_N\right) \tag{60}$$

is continuous rational function in $t \in \Delta_{N-1}$, since the denominator is always positive. □

Remark 52. There is an alternative way to see that the Donaldson-Futaki invariant is continuous which uses an intersection theoretic formula for the Donaldson-Futaki invariant [15, Proposition 6] which holds for normal test configurations. Assume that L_1 and L_2 are ample line bundles on X and $F_\bullet R_{L_1}$ and $G_\bullet R_{L_2}$ are admissible. The bigraded Proj

$$\mathscr{X} = \mathcal{P}roj_{\mathbb{A}^1}\, \mathcal{R}ees\, F_\bullet \left(R_{L_1} \otimes_{\mathbb{C}[t]} R_{L_2}\right) \tag{61}$$

with the Serre line bundle $\mathcal{O}(a,b)$ is a test configuration for the product $(X \times X, L_1^a \boxtimes L_2^b)$. Restricting $\mathscr{X}$ to the diagonal yields a test configuration $\mathscr{X}_{a,b}$ for $(X, L_1^a \otimes L_2^b)$. The filtration associated to $\mathscr{X}_{a,b}$ is equal to the filtration $\left(F_\bullet \otimes_{(a,b)} G_\bullet\right)(R_{L_1} \otimes R_{L_2})$ so the two test configurations are $\mathbb{G}_m$-equivariantly isomorphic.

If we assume that $\mathscr{X}$ is normal, the intersection theoretic formula for the Donaldson-Futaki invariant [15, p. 225] implies that the Donaldson-Futaki invariant is continuous in t.

The above argument generalises to weighted products of a finite collection of algebras.

We give a very simple example of a family of test configurations on a fixed polarised variety.

Example 53 (A combination of two simple test configurations on a ruled surface)**.** Let F and Q be very ample line bundles on a curve C of genus g and consider the projective bundle $\mathbb{P}(F \oplus Q)$ with its $\mathcal{O}(1)$-polarisation. Let α and β be the $\mathbb{G}_m$-actions which scale F and Q, respectively, with positive weight 1. The two $\mathbb{G}_m$-actions α and β determine filtrations

$$F \subset F \oplus Q \tag{62}$$

and

$$Q \subset F \oplus Q \tag{63}$$

and corresponding test configuration $\mathscr{Y}_F$ and $\mathscr{Y}_Q$ for $(\mathbb{P}(F \oplus Q), \mathcal{O}(1))$. The associated filtrations are discussed in more detail and generality in Section 7.

For any natural numbers a and b we define a test configuration of $\mathbb{P}(F \oplus Q)$ by inducing a $\mathbb{G}_m$-action on $\mathbb{P}\left(S^{a+b}(F \oplus Q)\right)$ and restricting to the image of $\mathbb{P}(F \oplus Q)$ under Veronese embedding of $\mathbb{P}(F \oplus Q)$. The filtration associated to this test configuration is generated by the grading on the vector bundle $S^{a+b}(F \oplus Q)$ given in Figure 1.

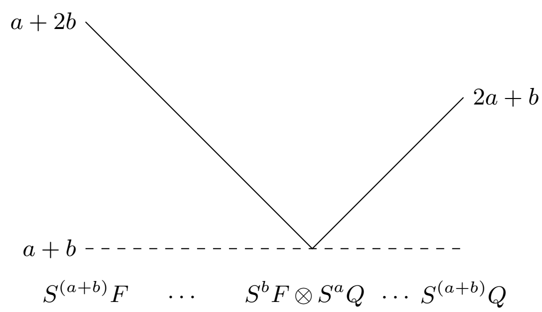

Figure 1: The t-grading on the $\mathcal{O}_{\mathbb{P}^1}$-module $S^{a+b}(F \oplus Q)$.

An elementary summation shows that the Donaldson-Futaki invariant of the test configuration

$a\tau_F + b\tau_Q$ is given by

$$\begin{aligned}\mathrm{DF}(a\tau_F + b\tau_Q) = \frac{a^3}{(a+b)^3}\,\mathrm{DF}(\mathscr{Y}_F) + \frac{b^3}{(a+b)^3}\,\mathrm{DF}(\mathscr{Y}_Q)\\ + \frac{a^2 b(\mu_F + 1 - g) + ab^2(\mu_G + 1 - g))}{2\mu_E^2 (a+b)^3}.\end{aligned} \tag{64}$$

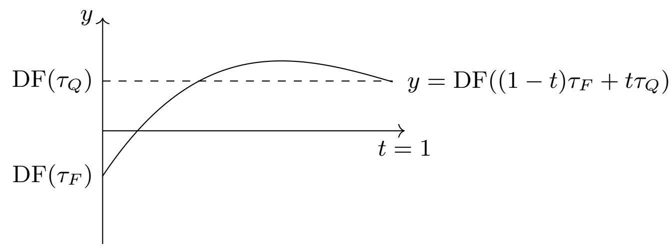

Figure 2: The Donaldson-Futaki invariant of $(1-t)\tau_F + t\tau_Q$ plotted against $t = \frac{b}{a+b}$ when $\mu_F = 2$, $\mu_Q = 1$ and $g = 2$ equals $\frac{1}{9}(-1 + 6t - 3t^2 - t^3)$.

For example, if $\mu_F = 2$ and $\mu_Q = 1$, we plot the Donaldson-Futaki invariant for different values of a and b in Figure 2. The code for repeating the calculation be found in [12, Ruled surface interpolations].

5 Okounkov bodies and the convex transform of a filtrations

In this section we describe the behaviour of the convex geometry associated to the variation of filtered linear series coming from the convex structure defined in Section 4. We give a brief review of Okounkov bodies and the convex transform associated to an admissible filtration. For more details, we refer to Lazarsfeld-Mustaţă [14], Boucksom-Chen [4], Witt-Nyström [29] and Székelyhidi [27].

Let X be a smooth complex projective variety and L a line bundle on X with ring of sections $R = \bigoplus_{k=0}^{\infty} H^0(X, L^k)$. Fix a base point $p \in X$ and holomorphic coordinates $z_1, \ldots, z_n$ centred around p. Given $f \in R_k$ we may write

$$f = s z_1^{r_1} \cdots z_n^{r_n}, \tag{65}$$

for some $(r_1, \ldots, r_n) \in \mathbb{Z}^n$, where s is a holomorphic function on a neighbourhood of p which does not vanish at p. We keep the base point and the choice of coordinates fixed throughout the section.

We define a function $\nu \colon R \to \mathbb{Q}^n$ by setting

$$\nu(f) = \frac{(r_1, \ldots, r_n)}{k} \tag{66}$$

for any such $f \in R_k$.

Definition 54. Define the *Okounkov body* of L by $\Delta(L) = \overline{\nu(R)} \subset \mathbb{R}^n$.

It is well known that $\Delta(L)$ is a convex set. Given an admissible filtration $F_\bullet R$, we define

$$R^{\leq t} = \bigoplus_{k=0}^{\infty} F_{\lfloor tk \rfloor} R_k. \tag{67}$$

This determines a closed convex subset $\Delta(L)^{\leq t} = \overline{\nu(R^{\leq t})}$.

Definition 55. Define the *convex transform* of $F_\bullet R$ to be

$$G(x) = \inf\{t \colon x \in \Delta(L)^{\leq t}\}. \tag{68}$$

If x is rational we have $G(x) = \inf \left\{ \frac{\operatorname{lev} f}{\deg f} : \nu(f) = x \right\}$. The extension to real numbers is obtained as the pointwise largest function which is lower semicontinuous and agrees with the restriction the subset $\Delta(L) \cap \mathbb{Q}^n$.

Suppose now that L_1 and L_2 are ample line bundles on X. Let $F^i_\bullet R_{L_i}$ be admissible filtrations for $i = 1, 2$ and let $G_i \colon \Delta(L) \to \mathbb{R}$ be the convex transforms of the two filtered algebras.

Let a and b be nonnegative integers such that there exists a surjective homomorphism

$$\psi \colon S = \bigoplus_{k=0}^{\infty} (R_{L_1})_{ak} \otimes (R_{L_2})_{bk} \longrightarrow \bigoplus_{k=0}^{\infty} H^0(X, (aL_1 + bL_2)^k). \tag{69}$$

for all $k > 0$. The ring $R_{aL_1+bL_2}$ is naturally filtered by the image of $(F^1_\bullet \otimes_{(a,b)} F^2_\bullet)S$. The Okounkov body $\Delta(aL_1 + bL_2)$ is contained in the Minkowski sum $a\Delta(L_1) + b\Delta(L_2)$.

Set

$$U = \left\{ (x, v) \in \mathbb{R}^{2n} \colon \frac{x}{2} + v \in a\Delta(L_1), \frac{x}{2} - v \in b\Delta(L_2) \right\} \tag{70}$$

and define a real valued function $\widehat{H} : U \to \mathbb{R}$ by setting

$$\widehat{H}_{a,b}(x, v) = aG_1(\frac{x + 2v}{2a}) + bG_2(\frac{x - 2v}{2b}). \tag{71}$$

Theorem 56. *The convex transform $G_{a,b}(x)$ of the weighted product filtration $(F^1_\bullet \otimes_{(a,b)} F^2_\bullet)\,(R_{L_1} \otimes R_{L_2})$ is equal to the minimiser*

$$H_{a,b}(x) = \min\nolimits_{v \in U} \widehat{H}_{a,b}(x, v) \tag{72}$$

restricted to the Okounkov body $\Delta(aL_1 + bL_2)$.

Proof. Let $G_{a,b}(x)$ be the convex transform of the filtration $(F_\bullet \otimes_{(a,b)} G_\bullet)(R \otimes S)$. We must show that $H_{a,b}(x) = G_{a,b}(x)$ for x in

$$\Delta(aL_1 + bL_2) \subset a\Delta(L_1) + b\Delta(L_2) \tag{73}$$

Let $x \in \Delta(aL_1 + bL_2) \cap \mathbb{Q}^n$ and let ν_i and $\nu_{a,b}$ denote the convex transforms of $F^i_\bullet$ and $F^1_\bullet \otimes_{(a,b)} F^2_\bullet$, respectively. We have

$$\begin{aligned} G_{a,b}(x) &= \inf\left\{\frac{\mathrm{lev}(f)}{k} : f \in (R_{aL_1+bL_2})_k \text{ and } \frac{\nu_{a,b}(f)}{k} = x\right\} \\ &= \inf\left\{\frac{\mathrm{lev}(g)+\mathrm{lev}(h)}{k} : g \in R_{akL_1}, h \in R_{bkL_2} \text{ and } (\psi \circ \nu_{a,b})(g \otimes h) = x\right\} \\ &\geq \inf\left\{aG_1(\nu_1(g)) + bG_2(\nu_2(h)) : g, h \text{ as above}\right\} \\ &\geq H_{a,b}(x). \end{aligned} \tag{74}$$

On the other hand, let $\epsilon > 0$ and fix y and z such that

$$H_{a,b}(x) \geq aG_1(y) + bG_2(z) - \epsilon. \tag{75}$$

There exists $k > 0$ such that we can find $g \in (R_{L_1})_{ak}$ and $h \in (R_{L_2})_{bk}$ such that

$$\begin{aligned} &\nu_1(g) = y, \quad \nu_2(h) = z \\ &\frac{\mathrm{lev}(g)}{ak} \leq G_1(y) + \epsilon, \quad \text{and} \quad \frac{\mathrm{lev}(h)}{bk} \leq G_2(z) + \epsilon, \end{aligned}$$

where $\nu_i : R_{L_i} \to \Delta(L_i)$ are the two valuations. We have

$$\begin{aligned} G_{a,b}(x) &\leq (\mathrm{lev}(g) + \mathrm{lev}(h))/k \\ &\leq aG_1(y) + bG_2(z) + (a+b)\epsilon && \text{by choice of } g \text{ and } h \\ &\leq H_{a,b}(x) + (a+b+1)\epsilon && \text{by choice of } y \text{ and } z. \end{aligned}$$

Letting ϵ tend to 0 yields

$$G_{a,b}(x) \leq H_{a,b}(x). \tag{76}$$

If x is irrational, the value of $G_{a,b}(x)$ is obtained as the infimum

$$\liminf_{\delta \to 0} \left\{G_{a,b}(x') : |x - x'| < \delta\right\}. \tag{77}$$

The same argument works in this case as well, bearing in mind that we may approximate the value of $G_{a,b}$ at x by $G_{a,b}(x')$ arbitrarily closely since $G_{a,b}(x)$ is convex and bounded from below. □

Remark 57. This result can easily be extended to convex combinations of arbitrary finite collections of test degenerations of X.

Remark 58. It is convenient to work instead with the $\mathbb{Q}$-line bundle $\frac{aL_1+bL_2}{a+b}$ and reparametrise the family of functions $H_{(a,b)}(x)$ as a function

$$H_t : \Delta((1-t)L_1 + tL_2) \to \mathbb{R}, \tag{78}$$

where t ranges over the unit interval. We go a step further and identify the range of H_t with a subset of

$$V(L_1, L_2) = \mathrm{Conv}\,(\Delta(L_1) \times \{0\}, \Delta(L_2) \times \{1\}) \subset \mathbb{R}^n \times [0,1]. \tag{79}$$

It would be interesting to know what kind of behaviour the function H_t can exhibit on $V(L_1, L_2)$. The variation of Okounkov bodies was studied by Lazarsfeld-Mustaţă [14, Section 4].

If X is toric, Okounkov bodies are a particularly powerful tool. The following examples use the theory of toric varieties. Briefly, the ring of sections of a polarised toric variety (X_Δ, L) corresponding to a polytope $\Delta = \Delta(L) \subset \mathbb{R}^n$, where $\mathbb{R}^n$ contains a fixed lattice $\mathbb{Z}^n$, is given by

$$R = \bigoplus_{k=1}^{\infty} \frac{\mathbb{Z}^n}{k} \cap \Delta. \tag{80}$$

Sections of $H^0(X, L^k)$ are identified with points

$$m/k = (m_1/k, \ldots, m_n/k) \tag{81}$$

in the polytope Δ, where m_i are integers. Multiplication of two sections x and y under this identification corresponds to taking their *Minkowski average* $(x+y)/2$ in Δ.

Example 59 (Convex combinations of toric filtrations.)**.** Let X be a toric variety with two line bundles L_1 and L_2 with section rings R and S isomorphic to the sets of rational points in $\Delta(L_1)$ and $\Delta(L_2)$, respectively. Let $G_1 : \Delta(L_1) \to \mathbb{R}$ and $G_2 : \Delta(L_2) \to \mathbb{R}$ be lower semicontinuous convex functions and define filtrations

$$F_i^f R_k = \mathrm{span}_{\mathbb{C}}\{x \in P/k : f(x) \leq i\}, \tag{82}$$

and

$$F_i^g S_k = \mathrm{span}_{\mathbb{C}}\{\beta \in Q/k : g(\beta) \leq i\}. \tag{83}$$

In this case the (a,b)-weighted Minkowski average

$$\mathcal{P} = \frac{a\Delta(L_1) + b\Delta(L_2)}{a+b}, \tag{84}$$

is precisely the Okounkov body of $\frac{aL_1 + bL_2}{a+b}$ in the appropriate sense for $\mathbb{Q}$-line bundles. The family of convex transforms

$$G_{a,b} \colon \mathcal{P} \to \mathbb{R} \tag{85}$$

now characterises the family of test degenerations determined by the weighted product by Donaldson's theory of toric test configurations [8]. Denote $G_t = \frac{G_{a,b}}{a+b}$, where $t = \frac{b}{a+b}$. Studying the behaviour of G_t as t changes may be a useful explicit way to study the variation of test configurations in the weighted product.

Example 60. Consider two $\mathbb{G}_m$-actions α and β on $\mathbb{P}^1 = \operatorname{Proj}\mathbb{C}[x,y]$ such that if (x/y) is a local coordinate, α scales (x/y) by weight c and β by $-d$. The filtrations $F^\alpha_\bullet$ and $F^\beta_\bullet$ defined by α and β, respectively, have linear convex transforms on the polytope $P = Q = [0,1]$. Rational points in $[0,1]$ correspond to monomials $x^p y^q$ by the bijection

$$x^p y^q \leftrightarrow p/(p+q). \tag{86}$$

It is straightforward to check, either from the definitions or by Theorem 56, that the convex transforms of $F^\alpha_\bullet$,$F^\beta_\bullet$ and $[F^\alpha_\bullet]+[F^\beta_\bullet]$ are

$$\begin{aligned} f_\alpha(x) &= 1 + cx, \\ f_\beta(x) &= 1 + d(1-x) \\ f_{\alpha\otimes\beta}(x) &= \max\{1 + c(x-1/2), 1 - d(x-1/2)\}, \end{aligned} \tag{87}$$

respectively. Geometrically, the corresponding degeneration splits $\mathbb{P}^1$ into two copies of $\mathbb{P}^1$ of equal volume intersecting at a fixed point of the $\mathbb{G}_m$-action. The $\mathbb{G}_m$-actions on the two components are given by scaling a local coordinate by the integers c and $-d$, respectively.

Example 61. Keep to the notation of Example 60, except now let $c = -d = 1$ and consider the (a,b)-diagonal product of filtrations

$$(F^\alpha_\bullet \otimes_{(a,b)} F^\beta_\bullet)(\mathbb{C}[x,y] \otimes_{\mathbb{C}} \mathbb{C}[x,y]) \tag{88}$$

for each pair of natural numbers (a,b). The total space of the toric family is, for each pair (a,b), a degeneration of a rational curve into a pair of intersecting curves of lower degree whose ratio of volumes is equal to t. As t approaches 0, the limiting convex function corresponds to the vector field β. This is also the natural limiting object in $\overline{Test(\mathbb{P}^1)}$.

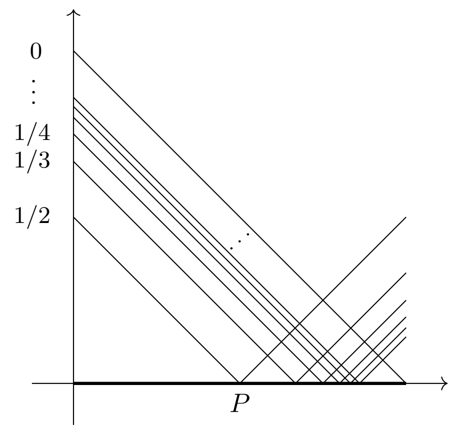

Figure 3: The convex functions corresponding to the product $a[F_\bullet^\alpha]+b[G_\bullet^\beta]$ in $\overline{Test(\mathbb{P}^1)}$ for different values of t, where we denote $t = b/(a+b)$.

6 Pullback test configurations

We fix a projective morphism $p\colon Y \to B$ and let L be an ample line bundle on B. In Section 3 we defined test configurations which are fibred over B in a $\mathbb{G}_m$-equivariant way. As a further application of the constructions of the previous sections, we construct test configurations of Y which are naturally fibred over a test configuration of B called *pullback test configurations.*

Let $F_\bullet R_L$ be an element of $Test(B)$. After replacing L with a power if necessary, we obtain an admissible filtration of $\mathcal{R}_L$, also denoted by $F_\bullet \mathcal{R}_L$. Let $\mathcal{L}$ be a relatively ample line bundle on Y and define a map

$$\Phi_{(a,b)} : Test(B) \to Test_B(Y) \tag{89}$$

by letting $\Phi(F_\bullet \mathcal{R}_L)$ be the the filtration

$$\bigoplus_{k=0}^{\infty} \mathcal{A}_{ak} \otimes F_\bullet L^{bk}. \tag{90}$$

Lemma 62. *The map Φ preserves admissible filtrations.*

Proof. This is a special case of Lemma 17. □

Definition 63. We say that $\Phi_{(a,b)}(F_\bullet \mathcal{R}_L)$ is the *pullback of $F_\bullet \mathcal{R}_L$ weight* (a,b).

Example 64 (Pullbacks of test configurations)**.** Assume that $F_\bullet R_L$ is a finitely generated admissible filtration and let $\mathscr{B}$ be the scheme $\operatorname{Proj} F_\bullet R_L$. Considering the algebra $\mathcal{R}ees_{\mathcal{O}_B}\,\Phi_{(a,b)}(F_\bullet \mathcal{R}_L)$ as a $\mathcal{O}_{\mathscr{B}}$-algebra determines a morphism

$$\mathscr{Y} = \mathcal{P}roj_B\, \mathcal{R}ees_{\mathcal{O}_B}\,\Phi_{(a,b)}(F_\bullet \mathcal{R}_L) \tag{91}$$

such that the diagram

$$\begin{tikzcd} \mathscr{Y} \arrow[r] \arrow[dr] & \mathscr{B} \arrow[d] \\ & \mathbb{A}^1 \end{tikzcd}$$

commutes.

Definition 65. Define the line bundle

$$\mathcal{L}_{a,b} = \mathcal{O}(a) \otimes p^* L^b \tag{92}$$

on $\mathcal{P}roj_B\, \mathcal{A}$. Alternatively, the line bundle $\mathcal{L}_{a,b}$ is the Serre line bundle on $\mathcal{P}roj\, (\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{R}_L)_{(a,b)}$. We have already seen in Lemma 33 that given a locally finitely generated p-test degeneration $G_\bullet \mathcal{A} \in Test_B(Y)$, the relative test configuration

$$\mathscr{Y} = \mathcal{P}roj_B\, (\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{R}_L)_{(a,b)} \tag{93}$$

is ample for $d \gg 0$. Denote the Serre line bundle on $\mathscr{Y}$ by $\mathscr{L}_{(a,b)}$. In particular, if $a = 1$ simply write $\mathscr{L}_{(a,b)} = \mathscr{L}_b$.

We give two examples of a nice phenomenon which happens with pullback test configurations for adiabatic polarisations. The first example, due to Stoppa [25], was already mentioned in Section **??**.

Example 66. Let $p\colon Y \to B$ be a blow up of a zero dimensional subscheme Z and $\mathcal{B}$ a test configuration for (B, L). Let $\mathscr{Y}$ be the pullback of $\mathcal{B}$ of weight $(1, m)$. Then the Donaldson-Futaki invariant of the test configuration $\mathrm{DF}(\mathscr{Y}, \mathscr{L}_m)$ is given by

$$\mathrm{DF}(\mathscr{Y}, \mathscr{L}_m) = \mathrm{DF}(\mathcal{B}) - Cm^{1-n} + O(m^{-n}), \tag{94}$$

where n is the dimension of B and C is a positive constant.

Similar results were also proved for slope stability by Ross and Thomas [20, Section 5.5], and later by Stoppa [26, Lemma 3.1].

The second example is due to Ross and Thomas [20, Section 5.4].

Example 67. Let $p\colon Y \to B$ be a projective bundle or a flag bundle and B' a subscheme of B. Let $\mathscr{Y}$ be a pullback test configuration with weight $(1, m)$ of the slope test configuration of $\mathcal{I}_{B'} \subset \mathcal{O}_B$ defined in Remark 34 with slope parameter 1. Then the leading term in $m \in \mathbb{N}$ of the Donaldson-Futaki invariant of the test configuration $\mathrm{DF}(\mathscr{Y}, \mathscr{L}_m)$ is given by

$$\mathrm{DF}(\mathscr{Y}, \mathcal{L}_m) = \mathrm{DF}(\mathcal{B}) + O(m^{-1}), \tag{95}$$

where $(\mathcal{B}, L)$ is the test configuration determined by the pullback of B'.

Ross and Thomas presented the calculation in the case of a projective bundle but the flag bundle case follows verbatim.

Remark 68. In the following we have various spaces of sections endowed with natural $\mathbb{G}_m$-actions. For each vector space we wish to have a succinct and obvious notation for the trace function defined on page **??**. Given a vector space V with a natural $\mathbb{G}_m$-action, we write the trace function simply as $\operatorname{tr} V$.

Remark 69. A product of two cscK polarised varieties (X_1, L_1) and (X_2, L_2) is cscK with respect to the product polarisation $L_1 \otimes L_2$. It is our hope that an algebraic proof of the K-stability of the polarisation $L_1 \otimes L_2$ would be found. The difficulty is having to consider test configurations which are not pullbacks from either X_1 or X_2. We believe it should not be necessary to consider these more complicated test configurations to decide whether $(X_1 \times X_2, L_1 \otimes L_2)$ is K-stable, in contrast with the example of an unstable product of two curves in [19].

Remark 70 (Toric bundles). There is a simple type of relative test configuration that has appeared in [1]. Let $\mathbb{E}$ be a principal $\mathrm{GL}(n, \mathbb{C})$-bundle over B and consider a torus bundle $\mathbb{T}$ in $\mathbb{E}$ with fibre $(\mathbb{G}_m)^{\times e}$. Then one may define a fibrewise orbit closure Y of $\mathbb{T}$ using the theory of toric varieties. The theory of toric test configurations developed in [8] generalises to this context and yields test configurations which intuitively degenerate fibres of the projection $Y \to B$ in a uniform way. The authors of [1] proved partial results about the extremal YTD correspondance for adiabatic polarisations on toric bundles constructed in this way.

We think of the test configurations defined in [1], which preserve the homotopy type of the associated principal bundle but degenerate the fibres of $p \colon Y \to B$, as complementary to the test configuration defined in [**?**]. We studied test configurations which changes the homotopy type of the associated principal $\mathrm{GL}(n, \mathbb{C})$-bundle but preserves the fibres of p.

In light of the previous remarks, we conclude that particularly on adiabatic polarisations of Y, there are two natural families of test configurations: ample p-test configurations and pullback test configurations. A perhaps naive conjecture we wish to make, motivated by known partial results on blowups, projective bundles, rigid toric bundles blowups and now flag bundles, is that these two test classes of test degenerations characterise the stability of adiabatic polarisations in the following sense.

Conjecture 1. *Let $p \colon Y \to B$ be a projective morphism with (B, L) a polarised variety and $\mathcal{L}_{(a,b)}$ as in Definition 65. Then there exists an integer $b_0 > 0$ such that the pair $(Y, \mathcal{L}_{(a,b)})$ is K-stable (K-polystable, K-semistable) for $b > b_0$ if and only if it is K-stable with respect to test configurations in $Test_B(Y, \mathcal{L}_{(a,b_0)})$ and pullback test configurations under the projection p with weight (a, b_0).*

Remark 71 (Some remarks about Conjecture 1). The hypothesis that projective morphism should be enough to yield the statement may be overenthusiastic as we have only studied very simple examples (flag bundles in [**?**] and certain closed immersions in [**?**]) in this work.

We also conjecture that the Conjecture 1 holds with admissible filtrations and $\overline{\mathrm{K}}$-stability in place of test configurations and K-stability.

Finally, an example in Ross [19] shows that the statement of the conjecture does not hold for arbitrary polarisations on Y.

7 Natural filtrations of shape algebras

Fix a coherent sheaf $\mathcal{E}$ with a subsheaf $\mathcal{F}$ on a scheme B, a partition λ with jumps given by r. Then we define a filtration $W_\bullet S_\lambda(\mathcal{E})$ which is generated by $\mathcal{F} \subset \mathcal{E}$ (cf. Definition 11 and Definition 18). The basic idea goes back to Griffiths, who defined a natural filtration of an exterior power of a vector bundle [10].

Example 72. The filtration of $S(\mathcal{E})$ generated by $\mathcal{F} \subset \mathcal{E}$ is given by

$$\begin{aligned} &\mathcal{F} \subset \mathcal{E} \oplus S^2\mathcal{F} \subset \mathcal{E} \oplus \mathcal{F} \cdot \mathcal{E} \oplus S^3\mathcal{F} \\ &\subset \mathcal{E} \oplus S^2\mathcal{E} \oplus \mathcal{F} \cdot S^2\mathcal{E} \oplus S^4\mathcal{E} \subset \cdots . \end{aligned} \tag{96}$$

Here we have used the notation $\mathcal{F} \cdot \mathcal{E}$ to mean tensors in $S^2\mathcal{E}$ which are in the image of the symmetrisation map $\mathcal{F} \otimes \mathcal{E} \to S^2E$. Note that the same filtration can be obtained from the filtration $\mathcal{I}_{\mathbb{P}\mathcal{F}} \subset \mathcal{O}_{\mathbb{P}\mathcal{E}}$ using Remark 34.

In general, the subsheaf $\mathcal{F} \subset \mathcal{E}$ generates a filtration

$$W_\bullet \mathcal{E}^\lambda = (W_\bullet S_\lambda(\mathcal{E}))_1, \tag{97}$$

which we write in terms of the factors of $\mathcal{F}$ and $\mathcal{E}$ in the tensor algebra $T(\mathcal{E})$ as

$$W_i \mathcal{E}^\lambda = c_\lambda \left(\mathcal{F}^{\otimes i} \otimes \mathcal{E}^{\otimes (l-i)} \right) \otimes_{\mathbb{C}[\mathfrak{S}_i] \times \mathbb{C}[\mathfrak{S}_{l-i}]} \mathbb{C}[\mathfrak{S}_l]. \tag{98}$$

Here c_λ is the Young symmetriser (cf. Definition **??**) and $\mathbb{C}[\mathfrak{S}_i]$ denotes the group algebra of the symmetric group, which acts on $T(\mathcal{E})$ by permuting the tensor factors. In other words, the module $W_i\mathcal{E}^\lambda$ is generated by tensors with at least i factors are contained in $\mathcal{F}$. The filtration in Equation (98) is a finite decreasing filtration and a simple change of indexing yields an increasing filtration which generates an admissible filtration of the algebra $S_\lambda(\mathcal{E})$. We call this filtration the $\mathcal{F}$-*weight filtration of* $S_\lambda(\mathcal{E})$ and denote it by $\widehat{W}^{\mathcal{F}}_\bullet S_\lambda(\mathcal{E})$. In contrast, we denote the filtration generated by the descending filtration of Equation (98) of increasing powers of $\mathcal{F}$ by $W_\bullet \mathcal{F} S_\lambda(\mathcal{E})$.

Remark 73. The test configuration determined by the subsheaf $\mathcal{F} \subset \mathcal{E}$ for flag bundles is not given by the theory of slope stability as it does in the case of projective bundles Example 72, but by a more complicated filtration of the structure sheaf $\mathcal{O}_{\mathcal{F}l_r(\mathcal{E})}$ (Remark 27 and Remark 34). This filtration is obtained from a flag of *relative Schubert varieties* determined by increasing incidence conditions with the subsheaf $\mathcal{F}$.

Example 74 (Computation of the weight function)**.** Consider a direct sum $\mathcal{F} \oplus \mathcal{Q}$ of coherent sheaves on B. We write

$$S_\lambda(\mathcal{F} \oplus \mathcal{Q})_k = (\mathcal{F} \oplus \mathcal{Q})^{k\lambda} = \bigoplus_{|\nu|+|\mu|=k|\lambda|} M^{k\lambda}_{\nu\mu} \mathcal{F}^\nu \otimes \mathcal{Q}^\mu \tag{99}$$

using the Littlewood-Richardson rule. We have

$$W_i E^{k\lambda} = \bigoplus_{|\nu| \le i} M^{k\lambda}_{\nu\mu} \mathcal{F}^\nu \otimes \mathcal{Q}^\mu. \tag{100}$$

We define the corresponding weight function

$$\begin{aligned} w(k) &= \sum_{i=0}^{\infty} i \left(\chi(W_i S_\lambda(\mathcal{F} \oplus \mathcal{Q})_k) - \chi(W_{i-1} S_\lambda(\mathcal{F} \oplus \mathcal{Q})_k) \right) \\ &= \sum_{i=0}^{\infty} i \bigoplus_{|\nu|=i} M^{k\lambda}_{\nu\mu} \mathcal{F}^\nu \otimes \mathcal{Q}^\mu \end{aligned} \tag{101}$$

This is the weight function which appeared in Lemma **??**.

Example 53 generalises to more general flag bundles.

Example 75 (A product of two simple filtrations of a shape algebra)**.** Let E be a vector bundle isomorphic to a direct sum of subbundles $F \oplus Q$. Let $\mathcal{A} = S_\lambda(E)$ be a shape algebra for $\mathcal{F}l_r(E)$ with a polarisation $\mathcal{L}_\lambda(A)$. Consider the two filtrations $W^F_\bullet \mathcal{A}$ and $W^Q_\bullet \mathcal{A}$. The filtration

$$F \otimes Q \subset F \otimes E \oplus Q \otimes E = S^2 E \tag{102}$$

generates the tensor product filtration $(W^F \otimes_{(1,1)} W^Q)(\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{A})$ of the $(1,1)$-diagonal of $\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{A}$ via the projection

$$\alpha \colon S_\lambda(S^2 E) \to S_{2\lambda}(E). \tag{103}$$

The kernel of α is a complicated object which can be described by decomposing the representation $S_\lambda(S^2 E)$ into irreducible representations. The composition of Schur functors is called *plethysm* [28, p. 63].

References

[1] V. Apostolov, D. Calderbank, P. Gauduchon, and C. Tønnesen-Friedman. Stability and extremal metrics on toric and projective bundles. 2006.

[2] V. Apostolov, D. Calderbank, P. Gauduchon, and C. T{onnesen-Friedman. Extremal Kähler metrics on projective bundles over a curve. *Advances in Mathematics*, 227(6):2385–2424, 2011.

[3] C. Arezzo and F. Pacard. Blowing up and desingularizing constant scalar curvature Kähler manifolds. *Acta mathematica*, 196(2):179–228, 2006.

[4] S. Boucksom and H. Chen. Okounkov bodies of filtered linear series. *Compositio Mathematica*, 147(04):1205–1229, 2011.

[5] N. Bourbaki. *Commutative algebra*, volume 8. Hermann Paris, 1972.

[6] G. Codogni and R. Dervan. Non-reductive automorphism groups, the Loewy filtration and K-stability, 2015.

[7] R. Dervan. Uniform stability of twisted constant scalar curvature Käahler metrics, 2014.

[8] S. Donaldson. Scalar curvature and stability of toric varieties. *Journal of Differential Geometry*, 62(2):289–349, 2002.

[9] D. Eisenbud. *Commutative Algebra: with a view towbard algebraic geometry*, volume 150. Springer Science & Business Media, 1995.

[10] P. Griffiths and J. Harris. Algebraic geometry and local differential geometry. In *Annales Scientifiques de l'École Normale Supérieure*, volume 12, pages 355–452. Société mathématique de France, 1979.

[11] R. Hartshorne. *Algebraic geometry.* Springer, 1977.

[12] A. Isopoussu. Code section. Personal homepage, March 2015.

[13] R. Lazarsfeld. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, volume 48. Springer, 2004.

[14] R. Lazarsfeld and M. Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér.*, (42), 2009.

[15] C. Li and C. Xu. Special test configuration and K-stability of Fano varieties. *Annals of mathematics*, 180(1):197–232, 2014.

[16] D. Mumford. *Stability of projective varieties.* L'Enseignement mathématique, 1977.

[17] Y. Odaka. A generalization of Ross-Thomas' slope theory. *Osaka J. Math*, 50(1), 2013.

[18] Y. Odaka. On parametrization, optimization and triviality of test configurations. *Proceedings of the American Mathematical Society*, 143(1):25–33, 2015.

[19] J. Ross. Unstable products of smooth curves. *Inventiones mathematicae*, 165(1):153–162, 2006.

[20] J. Ross and R. Thomas. An obstruction to the existence of constant scalar curvature Kähler metrics. *Journal of Differential Geometry*, 72(3):429–466, 2006.

[21] J. Ross and R. Thomas. A study of the Hilbert-Mumford criterion for the stability of projective varieties. *Journal of Algebraic Geometry*, 16(2):201–255, 2007.

[22] J. Ross and R. Thomas. Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics. *Journal of Differential Geometry*, 88(1):109–159, 2011.

[23] J. Ross and D. Witt-Nyström. Analytic test configurations and geodesic rays. *Journal of Symplectic Geometry*, 12(1):125–169, 2014.

[24] T. Stacks Project Authors. Stacks Project, April 2015.

[25] J. Stoppa. Unstable blowups. 2007.

[26] J. Stoppa and G. Székelyhidi. Relative K-stability of extremal metrics. *J. Eur. Math. Soc*, 13(4):899–909, 2011.

[27] G. Székelyhidi. Filtrations and test-configurations. *Mathematische Annalen*, pages 1–34, 2014.

[28] J. Weyman. *Cohomology of vector bundles and syzygies*, volume 149. Cambridge University Press, 2003.

[29] D. Witt-Nyström. Test configurations and Okounkov bodies. *Compositio Mathematica*, 148(06):1736–1756, 2012.