K-stability of relative flag varieties

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Abstract

We show that the relative flag variety associated to an unstable base variety is K-unstable, generalising the results of Ross and Thomas.

1 Introduction

Let E be a vector bundle of rank r_E on a polarised smooth complex variety (B, L) of dimension b, and $\mathcal{F}l_r(E)$ the flag bundle of r-quotients of E with projection p onto B. Also fix an ample line bundle $\mathcal{L}_\lambda(A) = \mathcal{L}_\lambda \otimes p^*A$ on $\mathcal{F}l_r(E)$, where λ is in $\mathcal{P}(r)$ and A is an ample line bundle on B.

In Section 2 we construct a test configuration $(\mathscr{Y}_\mathcal{F}, \mathscr{L}_\lambda(A))$ which we conjecture to be sufficient for detecting the K-instability of the flag bundle $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ assuming that the base B is stable.

From now on, we assume that λ is in $\mathcal{P}_\diamond(r)$. Section 4 calculates the Donaldson-Futaki invariant of $\mathscr{Y}_\mathcal{F}$ if we assume the base to be a curve.

Theorem 1. *Assume that B is a curve, E is ample and F is a subbundle of E whose degree is positive. There exists a test configuration $\mathscr{Y}_F$ for $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ such that*

$$\mathrm{DF}(\mathscr{Y}_F, \mathcal{L}_\lambda(A)) = C\left(\mu_E - \mu_F\right). \tag{1}$$

for some positive constant C depending on E, F, g and r.

In Section 5 we outline a similar calculation for adiabatic polarisations on a flag bundle over a base of arbitrary dimension.

Theorem 2. *Assume that $\mathcal{F}$ is a saturated torsion free subsheaf of E. Let L be an ample line bundle on B and assume that $A = L^m$. Then there exists an integer m_0 and a test configuration $\mathscr{Y}_\mathcal{F}$ for $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(L^m))$ such that for $m > m_0$ the Donaldson-Futaki invariant of $\mathscr{Y}_\mathcal{F}$ is given by*

$$\mathrm{DF}(\mathscr{Y}_\mathcal{F}, \mathscr{L}_\lambda(L^m)) = C\left(\mu_E - \mu_\mathcal{F}\right)\tfrac{1}{m} + O(\tfrac{1}{m^2}) \tag{2}$$

for some positive constant C depending on E, F, B and r.

These results immediately imply the stability statements of Theorem **??** and Theorem **??** from Section **??**.

Theorem 3 (The K-instability statements of Theorem A)**.** *Assume that B is a curve, E is an ample vector bundle on B and A is ample. If E is slope unstable and λ is in $\mathcal{P}_\diamond(r)$, then the flag bundle $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ is K-unstable. If E is not polystable, then the pair $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ is not K-polystable.*

Proof. Fix a destabilising subsheaf $\mathcal{F}$ of E with maximal slope. The saturation, which by definition has a torsion free quotient, also destabilises. Torsion free coherent sheaves on a curve are locally free, so we may assume that F is a subbundle. In particular E/F is locally free. The claim then follows from Theorem 1.

To prove the second assertion, let F be a subbundle of E with maximal slope such that $\mu(F) = \mu(E)$ and assume that F is not a direct summand. The scheme $\mathscr{Y}_F$ is smooth, so in particular it is normal. It follows that the test configuration is almost trivial only if it the total space $\mathcal{F}l_r(\mathcal{E})$ is isomorphic to $\mathcal{F}l_r(E) \times \mathbb{A}^1$ [11]. The two schemes $\mathcal{F}l_r(E)$ and $\mathcal{F}l_r(F \oplus E/F)$ are not isomorphic since it is possible to construct an isomorphism of underlying vector bundles from an isomorphism of flag bundles which preserves the polarisation. Therefore the bundle $\mathcal{F}l_r(E)$ is not K-stable. □

Theorem 4 (Theorem B)**.** *If E is slope unstable and λ is in $\mathcal{P}_\diamond(r)$, then there exists an m_0 such that the flag variety $\mathcal{F}l_r(E)$ of r-flags of quotients in E with the polarisation $\mathcal{L}_\lambda(L^m)$ is K-unstable for $m > m_0$.*

Proof. Follows immediately from Theorem 2. □

An identical argument to [10, Proposition 5.25] which will not be repeated here shows the following instability result which is also discussed in Example **??**.

Proposition 5. *If the base (B, L) is* strictly slope unstable *in the sense of [10, Definition 3.8], then there exists an $m_0 > 0$ such that $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(L^m))$ is K-slope unstable for $m > m_0$.*

2 Simple test configurations on flag bundles

In this section we define the relative test configuration $(\mathscr{Y}_\mathcal{F}, \mathscr{L}_\lambda(A))$. First, recall the following standard construction.

Definition 6 (The extension group of a coherent sheaf)**.** Let $\mathcal{F}$ and $\mathcal{Q}$ be coherent sheaves on B and let $p_1 \colon B \times \mathbb{A}^1 \to B$ be the first projection. An *extension* of $\mathcal{Q}$ by $\mathcal{F}$ is a coherent sheaf $\mathcal{E}'$ together with maps of $\mathcal{O}_B$-modules which fit the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E}' \to \mathcal{Q} \to 0. \tag{3}$$

Extensions are parametrised by the vector space $\mathcal{V} = \mathrm{Ext}^1(B, \mathcal{Q}, \mathcal{F})$ and there is a universal extension $\mathcal{U}$ on $B \times \mathcal{V}$ whose fibres are the corresponding extensions $\mathcal{E}'$. The sheaf $\mathcal{U}$ is naturally $\mathbb{C}^\times$-equivariant for the scaling action on $B \times \mathcal{V}$ which acts trivially on B.

Consider the reverse point of view where E is a fixed vector bundle fitting an exact sequence

$$0 \to \mathcal{F} \to E \to \mathcal{Q} \to 0. \tag{4}$$

Remark 7 (Turning off an extension). Let E be a locally free sheaf on B and $\mathcal{F}$ a quasicoherent subsheaf of E with quotient $\mathcal{Q}$. We abuse notation by writing $p_1^* E$ as $E[t]$ (we tacitly identify the algebra $\mathbb{C}[t]$ with the associated sheaf on $\mathbb{A}^1$), and identify $\mathcal{E}^{\mathcal{F}}$ as the subsheaf

$$\mathcal{E}^{\mathcal{F}} = p_1^*\mathcal{F} + tp_1^*E \subset p_1^*E = E[t]. \tag{5}$$

The sheaf $\mathcal{E}^{\mathcal{F}}$ is naturally isomorphic to the pullback of the universal extension under the inclusion

$$B \times \mathbb{A}^1 \to B \times \mathrm{Ext}^1(B, \mathcal{Q}, \mathcal{F}). \tag{6}$$

There is a natural $\mathbb{G}_m$-linearisation on $\mathcal{E}^{\mathcal{F}}$ of the standard $\mathbb{G}_m$-action on $B \times \mathbb{A}^1$. The fibre over $s \in \mathbb{A}^1$ of the sheaf $\mathcal{E}^{\mathcal{F}}$ is given by

$$\frac{\mathcal{E}^{\mathcal{F}}}{(t-s)\mathcal{E}^{\mathcal{F}}} \cong \begin{cases} E & \text{if } s \neq 0 \\ \mathcal{F} \oplus \mathcal{Q} & \text{if } s = 0. \end{cases} \tag{7}$$

In particular, the fibre of $\mathcal{E}$ over $s = 0$ is fixed by the $\mathbb{G}_m$-action, and so are all the fibres of $\mathcal{F} \oplus \mathcal{Q}$ over $B \times \{0\}$, so the linearisation is determined by a simple scaling action on the sections. Over the central fibre a section over an open set $U \subset B$ can be written as

$$\sigma = f + te + t\mathcal{E}^{\mathcal{F}}(U) \in \frac{\mathcal{E}^{\mathcal{F}}}{t\mathcal{E}^{\mathcal{F}}}(U) \tag{8}$$

Therefore we can write σ uniquely as $f + t\,(e + \mathcal{F}(U)) + t^2 E(U)$. The scaling action on $\mathbb{A}^1$ acts on the section t with weight -1.

We may renormalise the natural $\mathbb{G}_m$-linearisation on $\mathcal{E}^{\mathcal{F}}$ to scale sections of $\mathcal{F}$ with weight 1 and sections of $\mathcal{Q}$ with weight 0 over the central fibre. By Lemma **??**, we have an induced $\mathbb{G}_m$-action on the relative flag scheme

$$\mathcal{F}l_r(\mathcal{E}^{\mathcal{F}}) = \mathcal{P}roj_{B\times\mathbb{A}^1} S_\lambda(\mathcal{E}^{\mathcal{F}}) \tag{9}$$

with a natural linearisation on the Serre line bundle which we denote by $\mathscr{L}_\lambda$. The central fibre is isomorphic to $\mathcal{F}l_r(\mathcal{F} \oplus \mathcal{Q})$.

Let $\mathscr{L}_\lambda$ be the line bundle on $\mathscr{Y}_{\mathcal{F}} = \mathcal{F}l_r(\mathcal{E}^{\mathcal{F}})$ corresponding to a partition $\lambda \in \mathcal{P}(r)$. The $\mathbb{G}_m$-action on E induces a linearised action on $(\mathscr{Y}_{\mathcal{F}}, \mathscr{L}_\lambda)$. We extend this action trivially to any line bundle $\mathscr{L}_\lambda(f^*A)$, where $A \in \operatorname{Pic} B$ and $f\colon B \times \mathbb{A}^1 \to B$ is the projection. We will abuse notation by writing this line bundle simply as $\mathscr{L}_\lambda(A)$.

Claim 8. *Assume that B is a curve, E is an ample vector bundle on B and A is an ample line bundle on B. Let F be a subbundle of E of positive degree and maximal slope with quotient Q. Then $(\mathscr{Y}_F, \mathscr{L}_\lambda(A), \rho)$ is a test configuration for $(\mathcal{F}l_r(E), \mathcal{L}_\lambda)$.*

Proof. It suffices to show that the polarisation $\mathscr{L}_\lambda(A)$ is ample over the central fibre. Since E is ample, we may assume that $A = \mathcal{O}_B$. By Proposition **??** it suffices to show that $F \oplus Q$ is ample.

The bundle E/F is ample since it is a quotient of an ample bundle. The subbundle F has positive degree and it is stable so it is ample by [5, Section 2]. Therefore the Schur power $(F \oplus Q)^\lambda$ is ample by Proposition **??**, which proves the claim. □

Remark 9. We fully expect the statement of Claim 8 to be true if F is as above and we only assume $\mathcal{L}_\lambda(A)$ to be ample.

Claim 10. *Let L be an ample line bundle on B. Then the pair $(\mathscr{Y}_F, \mathscr{L}_\lambda(L^m), \rho)$ is a test configuration for $m \gg 0$.*

Proof. This follows immediately from [6, Proposition 7.10]. □

We call the $\mathbb{G}_m$-linearised pair $(\mathcal{F}l_r(E), \mathscr{L}_\lambda(A))$ the *simple test configuration induced by $\mathcal{F}$.*

Assume that the scheme $(\mathscr{Y}_\mathcal{F}, \mathscr{L}_\lambda(A))$ is a test configuration and let $h(k)$ and $w(k)$ be the Hilbert and weight polynomials. Let p_1 and p_2 be the two projection of the product $B \times \mathbb{P}^1$ and define the vector bundle

$$\widetilde{E} = p_1^* F \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1) \oplus p_1^* Q. \tag{10}$$

We write the vector bundle $\widetilde{E}$ simply as $\widetilde{E} = F(1) \otimes Q$.

Lemma 11. *The weight function $w(k)$ of the action ρ and the Hilbert function $h(k) = h^0(\mathcal{F}l_r(E), \mathcal{L}(A)^k)$ satisfy the identity*

$$w(k) + h(k) = \chi(B \times \mathbb{P}^1, \widetilde{E}^\lambda \otimes p_1^* A). \tag{11}$$

Proof. Assume first of all that $A = \mathcal{O}_B$. By the Littlewood-Richardson rule (see [14, (2.3.1) Proposition]) we have the decomposition

$$\widetilde{E}^\lambda = \bigoplus_{\nu,\mu} (F(1)^\nu \otimes Q^\mu)^{\oplus M^\lambda_{\nu\mu}}, \tag{12}$$

where the sum is over all partitions ν and μ whose sizes sum up to the size of λ and the coefficient $M^\lambda_{\nu,\mu}$ is the *Littlewood-Richardson coefficient.* Using the Künneth formula, Riemann-Roch on $\mathbb{P}^1$ and additivity of the Euler characteristic we see that

$$\begin{aligned}
\chi(B \times \mathbb{P}^1, \widetilde{E}^\lambda) &= \sum_{\nu,\mu,\lambda} M^\lambda_{\nu,\mu} \chi(B \times \mathbb{P}^1, F^\nu \otimes Q^\mu \otimes \mathcal{O}_{\mathbb{P}^1}(|\nu|)) \\
&= \sum_{\nu,\mu,\lambda} (|\nu| + 1) M^\lambda_{\nu,\mu} \chi(B, F^\nu \otimes Q^\mu) \\
&= \chi(B, E^\lambda) + \sum_{|\nu|+|\mu|=|\lambda|} |\nu| \chi(B, (F^\nu \otimes Q^\mu)^{\oplus M^\lambda_{\nu,\mu}}).
\end{aligned} \tag{13}$$

Assuming that the vector bundles $\widetilde{E}^\lambda$ and E^λ are ample, the weight $w(k)$ is given by

$$w(k) = \sum_{|\nu|+|\mu|=|\lambda|} |\nu| h^0 \left(B, (F^\nu \otimes Q^\mu)^{\oplus M^\lambda_{\nu,\mu}}\right). \tag{14}$$

Finally, the calculation works verbatim if the bundle A is nontrivial. □

Using Lemma 11 we can calculate both the Hilbert and the weight polynomials using the Hirzebruch-Riemann-Roch formula. For the former, we have

$$h(k) = \int_B \operatorname{ch}(E^{k\lambda}) \operatorname{ch}(A) \operatorname{Td}_B, \tag{15}$$

and similarly for the latter, we have

$$w(k) = \int_{B\times\mathbb{P}^1} \operatorname{ch}(\widetilde{E}^{k\lambda}) \operatorname{ch}(A) \operatorname{Td}_{B\times\mathbb{P}^1} - h(k). \tag{16}$$

There exist integers a_0, a_1, b_0 and b_1 so that we can write

$$\chi(B, E^{k\lambda}) = \operatorname{rank} E^{k\lambda} \left(a_0 k^b + a_1 k^{b-1} + O(k^{b-2})\right) \tag{17}$$

and

$$\chi(B\times\mathbb{P}^1, \widetilde{E}^{k\lambda}) = \operatorname{rank} E^{k\lambda} \left(b_0 k^{b+1} + b_1 k^b + O(k^{b-1})\right). \tag{18}$$

The common factor cancels and we get

$$\operatorname{DF}\left(\mathscr{Y}_{\mathcal{F}}, \mathscr{L}_\lambda(A)\right) = \frac{b_0 a_1 - b_1 a_0 + a_0^2}{a_0^2} \tag{19}$$

for the Donaldson-Futaki invariant.

The Chern classes of the twisted bundle $\widetilde{E}$ appearing in Equations (17) and (18) are given by the following Lemma.

Lemma 12. *Let $\widetilde{E}$ be the vector bundle defined in Equation* (10) *and* $\mathbf{h}$ *is the fibre of a point under* p_2. *We have*

$$\begin{aligned}
h_2(\widetilde{E}) &= r_F p_1^* c_1(E)\mathbf{h} + p_1^* c_1(F)\mathbf{h} + p_1^* h_2(E) + \frac{r_F(r_F+1)\mathbf{h}^2}{2}\\
c_2(\widetilde{E}) &= r_F p_1^* c_1(E)\mathbf{h} - p_1^* c_1(F)\mathbf{h} + p_1^* c_2(E) + \frac{r_F(r_F-1)\mathbf{h}^2}{2}\\
c_1(\widetilde{E}) &= p_1^* c_1(E) + r_F\mathbf{h}\\
A_2(\widetilde{E}) &= -\frac{r_F}{r_E+1}\left(\frac{p_1^* c_1(E)\mathbf{h}}{r_E} - \frac{p_1^* c_1(F)\mathbf{h}}{r_F}\right) + Z
\end{aligned} \tag{20}$$

where Z is contained in the image of p_1^* and the class $A_2(\widetilde{E})$ is defined in Lemma 18.

Proof. The proposition follows by direct computation from the Whitney sum formula [3, Theorem 3.2] and the general fact that we have

$$c_k(\mathcal{F} \otimes L) = \sum_{j=0}^{k} \binom{r-i+j}{j} c_{k-j}(\mathcal{F}) c_1(L)^j \tag{21}$$

for any locally free sheaf $\mathcal{F}$ and line bundle L [3, Example 3.2.2]. Alternatively, one may get the result using the splitting principle. □

Remark 13 (Optimal test configurations). Before proceeding with the proofs of Theorems 1 and 2, we make a naive but natural conjecture to make about the optimality of the test configuration $\mathscr{Y}_{\mathcal{F}}$. Assume that B is K-stable and $\mathcal{F}$ has maximal slope in the set of torsion free subsheaves of E. We conjecture that the test configuration $\mathscr{Y}_{\mathcal{F}}$ is a *maximally destabilising* test configuration of $(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A))$ in the sense that the quantity $\frac{\mathrm{DF}(\mathscr{Y})}{\|\mathscr{Y}\|}$ is bounded below by $\frac{\mathrm{DF}(\mathscr{Y}_{\mathcal{F}})}{\mathscr{Y}_{\mathcal{F}}}$.

Optimality of test configurations in this sense was studied by Székelyhidi in the case of toric varieties [13]. The difficulty in the general case stems from the difficulty of parametrising the collection of test configurations.

3 A formula for the Chern character of a Schur power

This section is entirely devoted to a technical result used in the computation of the weight polynomial of a flag bundle. We let r and λ be such that $\lambda \in \mathcal{P}(r)$ throughout. We also fix a smooth proper scheme B of dimension b and a vector bundle E of rank r_E. Let p be the projection $p\colon \mathcal{F}l_r(E) \to B$.

Of independent interest would be finding a more general and more elegant formulation for Theorem 16 (Theorem **??**), which gives a formula for the second order asymptotics of the polynomial $\operatorname{ch} E^{k\lambda}$ under certain hypotheses. Laurent Manivel has previously calculated the highest order term in [9, Section 3]. Background on Chern classes can be found in the seminal work of Grothendieck [**?**].

If P is a symmetric polynomial and E is a vector bundle with Chern roots $x_1, \ldots, x_{r_E}$, we write $P(E) = P(x_1, \ldots, x_{r_E})$. On the other hand it also makes sense to consider the polynomial P on the algebra generated by line bundles on a variety and operations defined by direct sums and tensor products. In this case we write $P(L_1, \ldots, L_{r_E})$ for the resulting vector bundle, not to be confused with $P(E)$, which is a cohomology class.

Let

$$c_r(x_1, \ldots, x_{r_E}) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le r_E} x_{i_1} \cdots x_{i_r} \tag{22}$$

denote the rth elementary symmetric polynomial in $x_1, \ldots, x_{r_E}$. Similarly we have the complete symmetric polynomial

$$h_r(x_1, \ldots, x_{r_E}) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r \le r_E} x_{i_1} \cdots x_{i_r}. \tag{23}$$

Recall that Schur polynomials are a basis of the algebra of symmetric funtion, which appear naturally when computing the cohomology of Schur powers of vector bundles. We define Schur polynomials by using the Giambelli formula [4, Appendix A] as

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1\le i,j\le l} \tag{24}$$

associated to a partition λ. In particular, $s_{(k)} = h_k$ and $s_{1^k} = c_k$.

Definition 14. Define the *canonical partition* $\sigma = \sigma_{r_E, r}$ depending on the parameter r by

$$\sigma_i = r_E + l(\lambda) - r^+(i) - r^-(i) \tag{25}$$

where $r^+(i)$ is the smallest integer in r satisfying $r^+(i) \ge i$ and $r^-(i)$ the largest integer in r satisfying $r^-(i) < i$.

Example 15 (The canonical bundle of a Grassmannian)**.** Consider the Grassmannian case $r = (p)$, where $1 \le p < r_E$. Now the canonical partition σ is the constant partition (r_E^p), which corresponds to the r_Eth multiple of the hyperplane bundle in the case $p = 1$. Note that the relative canonical bundle of $\mathbb{P}E$ over B is the dual of the corresponding line bundle $\mathcal{L}_\sigma$.

Theorem 16. *Let E be a vector bundle of rank E and λ a partition whose jumps are given by r. Assume that λ satisfies at least one of the following conditions*

- $l(\lambda) \le 4$
- $\lambda = t\sigma_{r_E, r}$ *for some* $t \in \mathbb{Q}$ *and* $r_E > r_c$.

Then there exist polynomials $B_i(E, \lambda) \in \mathbb{Q}[\lambda_1, \dots \lambda_l, c_1(E), \dots, c_{r_E}(E)]$ such that

$$\operatorname{ch} E^\lambda = \operatorname{rank} E^\lambda \left(1 + B_1(E,\lambda) + B_2(E,\lambda) + \cdots + B_n(E,\lambda)\right) \tag{26}$$

where $B_i(E,\lambda)$ is homogeneous of degree i as an element of the Chow ring of X and of degree i in the λ_i. The polynomials $B_1(E,\lambda)$ and $B_2(E,\lambda)$ are given by

$$B_1(E,\lambda) = \frac{c_1(\lambda)c_1(E)}{r_E} \tag{27}$$

and

$$\begin{aligned} B_2(E,\lambda) \equiv_1 & \frac{h_2(\lambda)h_2(E)}{r_E(r_E+1)} + \frac{c_2(\lambda)c_2(E)}{r_E(r_E-1)} \\ & + \frac{r_E c_1(\lambda) - \sum_i (2i-1)\lambda_i}{2}\left(\frac{h_2(E)}{r_E(r_E+1)} - \frac{c_2(E)}{r_E(r_E-1)}\right) + O(1). \end{aligned} \tag{28}$$

where $O(1)$ denotes a term independent of λ. By the equivalence $\equiv_1$ we mean the following: If U and V are k-cycles in B, then $U \equiv_1 V$ if $c_1(A)^{n-k}.(U - V)$ is equal to 0 for all line bundles $A \in \operatorname{Pic} B$.

It is straightforward to check in cases which yield to computer analysis that it is not necessary to assume $\diamond$ for the identity in Equation (**??**) to hold, but we were unable to find a proof in the general case. Under the assumption $\diamond$, we prove the statement using the following determinantal identity, which the author learned from a paper [2] pointed out by Will Donovan.

Lemma 17 (Determinantal identity)**.** *Let E be a vector bundle of rank r_E and λ a partition of length l. The Chern character of a Schur power of E is*

$$\mathrm{ch}E^{\lambda} = \det\left(\mathrm{ch}(S^{\lambda_i+j-i}E)\right)_{i,j} \tag{29}$$

Proof. By the splitting principle [3, Remark 3.2.3] we may assume that $E = L_1 \oplus \cdots \oplus L_{r_E}$. Let p be a polynomial function on the set of factors $L_1, \ldots, L_{r_E}$ with integral coefficients a_I for $I = (i_1, \ldots, i_{r_E})$. We denote

$$p(L_1, \ldots, L_{r_E}) = \bigoplus_I \left(L_1^{i_1} \otimes \cdots \otimes L_{r_E}^{i_{r_E}}\right)^{\oplus a_I}, \tag{30}$$

Schur powers of decomposable vector bundles can be expressed in as

$$E^{\lambda} = s_{\lambda}(L_1, \ldots, L_{r_E}), \tag{31}$$

which we expand as a determinant using Equation (24)

$$s_{\lambda}(L_1, \ldots, L_{r_E}) = \det\left(h_{\lambda_i+j-i}(L_1, \ldots, L_{r_E})\right)_{i,j}. \tag{32}$$

Taking Chern characters on both sides completes the proof of the Lemma. □

Lemma 18. *Let E be a vector bundle of rank r_E. The Chern character of the bundle S^kE is*

$$\binom{k+r_E-1}{r_E}\left(1 + \frac{c_1(E)}{r_E}k + A_1(E)k^2 + A_2(E)k + Z\right), \tag{33}$$

where $A_1(E), A_2(E) \in \mathbb{Q}[x_1 \ldots x_{r_E}]$ are given by

$$A_1(E) = \frac{h_2(E)}{r_E(r_E+1)}, \tag{34}$$

$$A_2(E) = \frac{r_E-1}{2}\left(\frac{h_2(E)}{r_E(r_E+1)} - \frac{c_2(E)}{r_E(r_E-1)}\right) \tag{35}$$

and Z is a sum of terms of Chow degree 3 and higher.

Proof. Recall the definition of the monomial symmetric function m_μ of partition μ of length at most n. Given variables $y = (y_1, \ldots, y_n)$ we set

$$m_{\mu}(y) = \sum_{\sigma \in \mathfrak{S}_n} y_{\sigma(1)}^{\mu_1} \cdots y_{\sigma(n)}^{\mu_n}. \tag{36}$$

We have

$$\begin{aligned}
\mathrm{ch}(S^kE) &= \mathrm{ch}\, h_k(E) \\
&= \mathrm{ch} \sum_{\mu} m_\mu(E) \\
&= \sum_{\mu} (1 + \mu_1 x_1 + \mu_1^2 x_1^2/2 + \cdots) \cdot \cdots \cdot (1 + \mu_{r_E} x_{r_E} + \mu_{r_E}^2 x_{r_E}^2/2 + \cdots)
\end{aligned} \tag{37}$$

where the sum is over all r_E-tuples that sum to k. The rest of the computation is an elementary summation. The Chow-degree one part of $\mathrm{ch}(S^kE)$ is

$$\mathrm{ch}(S^kE)_1 = \mathrm{rank}\left(S^kE\right) \frac{c_1(E)}{r_E}, \tag{38}$$

where

$$\mathrm{rank}\left(S^kE\right) = \binom{k + r_E - 1}{r_E - 1}. \tag{39}$$

The degree two term can be written as

$$\sum_{i=1}^{k} \sum_{j=1}^{k-i} ij \binom{r_E - 3 + k - i - j}{r_E - 3} \sum_{l<m}^{r_E} x_l x_m + \sum_{i=1}^{k} i^2 \binom{r_E - 2 + k - i}{r_E - 2} \sum_{l=1}^{r_E} x_l^2/2, \tag{40}$$

which using the combinatorial identities proved in the appendix simplifies to

$$\frac{(k + r_E - 1)!}{(k-2)!(r_E + 1)!} \sum_{m<l}^{r_E} x_m x_l + \frac{(r_E + 2k - 1)(k + r_E - 1)!}{(k-1)!(r_E+1)!} \sum_{m=1}^{r_E} x_m^2/2. \tag{41}$$

Picking out the rank r_{S^kE} of S^kE as a common factor yields

$$\mathrm{ch}_2(S^kE) = r_{S^kE} \left(\frac{k(k-1)}{r_E(r_E+1)} \sum_{m<l}^{r_E} x_m x_l + \frac{2k^2 + k(r_E - 1)}{r_E(r_E+1)} \sum_{m} x_m^2/2 \right) \tag{42}$$

Recall that the Chern classes of E, when written in terms of the x_i, are

$$c_1(E)^2 = h_2(E) + c_2(E) = \sum_{m=1}^{r_E} x_m^2 + 2 \sum_{m<l}^{r_E} x_m x_l \tag{43}$$

and

$$c_2(E) = \sum_{m<l}^{r_E} x_m x_l. \tag{44}$$

Thus we have

$$\mathrm{ch}\left(S^kE\right) = \mathrm{rank}(S^kE) \left(1 + \frac{c_1(E)}{r_E} k + A_1(E) k^2 + A_2(E) k + Z \right), \tag{45}$$

where

$$A_1(E) = \frac{h_2(E)}{r_E(r_E+1)}, \tag{46}$$

$$A_2(E) = \frac{(r_E-1)c_1(E)^2}{2r_E(r_E+1)} - \frac{c_2(E)}{r_E+1} = \frac{r_E-1}{2}\left(\frac{h_2(E)}{r_E(r_E+1)} - \frac{c_2(E)}{r_E(r_E-1)}\right) \tag{47}$$

and Z is a sum of terms of Chow degree 3 and higher □

Remark 19. The length of a partition λ whose jumps are given by r is the largest integer r_c in r.

Proposition 20. *Theorem 16 holds for partitions up to length 4.*

Proof. This is an easy calculation for a computer using Lemma 18 and Lemma 17 [7, Calculation of Chern classes for Schur powers]. □

Remark 21 ([9, Section 3]). Alternatively one may expand the Chern character of S^kE as

$$\sum_{p,q} x^p \prod_{i=1}^{r} \frac{a_{p_i,q_i}}{p_i!}\binom{k+r_E-1+|q|}{r_E-1+|p|} \tag{48}$$

where p,q range over r-tuples of nonnegative integers and $a_{i,j}$ is the jth coefficient of the ith Euler polynomial [9, Proposition 2.2]. This way the existence of claimed decomposition

$$\mathrm{ch}(S^kE) = \mathrm{rank}(S^kE)A(k) \tag{49}$$

is clear for higher degree terms as well. The determinantal identity implies that we have

$$\mathrm{ch}(E^\lambda) = \sum_{p_i,q_j\in\mathbb{N}^{r_E}} \frac{x^{p_1+\cdots+p_l}}{p_1!\cdots p_l!} a_{p_1,q_1}\cdots a_{p_l,q_l} \det\left(\binom{r_E+\lambda_i+|q_i|-i+j-1}{r_E+|p_i|-1}\right)_{1\le i,j\le l} \tag{50}$$

Let $p:\mathbb{P}E\to X$ denote the projection. It is well known that we have the pushforward formula

$$\int_{\mathbb{P}E} p_*c_1\left(\mathcal{O}_{\mathbb{P}E}(1)\right)^{n+r-1} = \int_X h_n(E). \tag{51}$$

This formula generalises to the following theorem by Laurent Manivel.

Theorem 22 ([9, Proposition 3.1])**.** *Let λ be a partition whose jumps are given by r and $m\mathbb{Z}_{\ge 0}$. Then we have*

$$p_*\frac{c_1(\mathcal{L}_\lambda)^{N+m}}{(N+m)!} \equiv_1 C_{\lambda,r_E} \sum_{|\mu|=m,l(\mu)\le l(\lambda)} \frac{s_\mu(\lambda)s_\mu(E)}{\prod_{k=1}^{l(\lambda)}(r_E+\mu_k-k)!}, \tag{52}$$

where $C_{\lambda,r_E} = \prod_{i=1}^{l(\lambda)}(s^+(i)-i)!\prod_{\lambda_i>\lambda_j}(\lambda_i-\lambda_j)$. For $m=n$ we have equality of cycles, while for $m<n$, the relation $\equiv_1$ is the one defined in Theorem 16

Remark 23. The result stated in [9] actually claims equality at the level of cycle classes. As we were unable to reproduce the details which were left for the reader in the paper, we state a slightly weaker result, but this is enough for our purposes.

Remark 24. Although the highest order term of each $B_i(E,\lambda)$ is a symmetric function with respect to the λ, this is not the case for the lower order terms, or indeed for the entire Chern character.

Remark 25. In particular, Theorem 22 computes the leading coefficient

$$D_{\lambda,r_E} := \frac{C_{\lambda,r_E}}{\prod_{i=1}^{l(\lambda)}(r_E - i)!} \tag{53}$$

of the Hilbert polynomial of a fibre $\pi^{-1}(x)$ for any $x \in B$.

Remark 26. We can write the line bundle $\mathcal{L}_\sigma$ in terms of the tautological subbundles as

$$\bigotimes_{i=1}^{c} (\det \mathcal{R}_i^*)^{r_{i+1}-r_{i-1}}. \tag{54}$$

Lemma 27 (Canonical bundle of the flag variety)**.** *The canonical class of $\mathcal{F}l_r(E)$ is*

$$c_1(\mathcal{L}_{-\sigma} \otimes p^* \left(K_B \otimes \det E^{l(\sigma)}\right)), \tag{55}$$

where σ is the canonical partition defined Definition 14 and $\mathcal{L}_{-\sigma}$ denotes the dual of $\mathcal{L}_\sigma$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{V}_{\mathcal{F}l_r(E)} \longrightarrow \mathcal{T}_{\mathcal{F}l_r(E)} \longrightarrow \mathcal{H}_B \longrightarrow 0 \tag{56}$$

where $\mathcal{V}_{\mathcal{F}l_r(E)}$ is the relative tangent bundle of the fibration $\mathcal{F}l_r(E) \to B$, $\mathcal{T}_{\mathcal{F}l_r(E)}$ is the tangent bundle and $\mathcal{H}_B$ is isomorphic to the pullback of the tangent bundle of the base B. The relative tangent bundle $\mathcal{V}_{\mathcal{F}l_r(E)}$ has a filtration

$$0 \subset F_1 \subset \cdots \subset F_N \subset \mathcal{V}_{\mathcal{F}l_r(E)} \tag{57}$$

such that

$$\bigoplus_{i=1}^{N} F_{i+1}/F_i = \bigoplus_{1\le i<j\le c} \mathcal{Q}_i \otimes \mathcal{Q}_j^* \tag{58}$$

This can be seen by successive fibrations by bundles of r'-flags, where r' is a subset of r [8]. We have

$$\det(\mathcal{V}_{\mathcal{F}l_r(E)})^* \cong \det\left(\bigoplus_{1\le i<j\le c} \mathcal{Q}_i \otimes \mathcal{Q}_j^*\right)^* \tag{59}$$

Denote $\det \mathcal{R}_i^* = L_i$ and define

$$A(k) := \det\left(\bigoplus_{1\le i<j\le k+1} \mathcal{Q}_i \otimes \mathcal{Q}_j^*\right) = \det\left(\bigoplus_{1\le i<j\le k} \mathcal{R}_i^*/\mathcal{R}_{i-1}^* \otimes \mathcal{R}_j/\mathcal{R}_{j-1}\right). \tag{60}$$

We expand the determinant of the vector bundle of Equation (58) as

$$A(c) = \det\left(\bigoplus_{1\le i<j\le c+1} \mathcal{Q}_i \otimes \mathcal{Q}_j^*\right) = \det\left(\bigoplus_{1\le i<j\le c} \mathcal{R}_i^*/\mathcal{R}_{i-1}^* \otimes \mathcal{R}_j/\mathcal{R}_{j-1}\right). \tag{61}$$

This is convenient to write in additive notation as

$$\sum_{1\le i<j\le c+1} \left(-(r_i - r_{i-1})\left(L_j - L_{j-1}\right) + (r_j - r_{j-1})\left(L_i - L_{i-1}\right)\right). \tag{62}$$

We have

$$A(k) - A(k-1) = r_k L_{k-1} - r_{k-1} L_k. \tag{63}$$

for any $1 \le k \le c$. Therefore, we can see that the sum in Equation 62 telescopes and we find

$$A(c) = \sum_{i=1}^{c} (r_{i+1} - r_{i-1}) L_i - r_c L_{c+1}. \tag{64}$$

Finally, the identity

$$K_{\mathcal{F}l_r(E)} = -A(c) + p^* K_B, \tag{65}$$

follows from Equation 56. This completes the proof of the Lemma. □

Lemma 28. *Let r be an increasing sequence of c positive integers. Then $\sigma = \sigma_{r_E,r}$ is a partition of length r_c with $r_c < r_E$. We have*

$$|\sigma| = r_E r_c, \tag{66}$$

$$\sum_{i=1}^{r_c} (2i-1)\sigma_i = r_c^2 r_E - \sum_{i=1}^{c-1} r_i r_{i+1}(r_{i+1} - r_i), \tag{67}$$

and

$$h_2(\sigma) = \frac{1}{2}\left(r_c {r_E}^2 (r_c + 1) + \sum_{i=1}^{c-1} r_i r_{i+1}(r_{i+1} - r_i) \right), \tag{68}$$

Proof. The proof is a direct calculation. We prove the third identity, which is marginally more difficult than the first two. First notice that given an integer n and an l-tuple λ, we have

$$h_2(n + \lambda) = \frac{l(l+1)}{2} n^2 + (l+1)n|\lambda| + h_2(\lambda). \tag{69}$$

where n is considered to be the constant l-tuple $(n, \dots, n)$. Applying this in the case $n = r_E + r_c$ and $\lambda = -(r^+ + r^-)$ it suffices to show that

$$h_2(r^+ + r^-) = \frac{1}{2}\left(r_c^3 (r_c + 1) + \sum_{i=1}^{c-1} r_i r_{i+1}(r_{i+1} - r_i) \right). \tag{70}$$

This is proved by induction. Let s be the tuple $(r_1, \dots, r_{c-1})$. We then have

$$\begin{aligned} h_2(r^+ + r^-) - h_2(s^+ + s^-) &= (r_c + r_{c-1})^2 (r_c - r_{c-1})(r_c - r_{c-1} + 1)/2 \\ &\quad + \sum_{i=1}^{c-1} (r_i - r_{i-1})(r_i + r_{i-1})(r_c - r_{c-1})(r_c + r_{c-1}) \\ &= r_c^3 (r_c + 1)/2 + r_{c-1}^3 (r_{c-1} + 1)/2 + r_c r_{c-1}(r_c - r_{c-1})/2 \end{aligned} \tag{71}$$

from which the claim follows. □

Let $N_{r_E,r}$ denote the relative dimension of a bundle of r-flags, given by

$$N_{r_E,r} = \sum_{i=1}^{c} r_i(r_{i+1} - r_i), \tag{72}$$

with the convention $r_{c+1} = r_E$.

Proof of Theorem 16. Retain the notation in the statement of the Theorem and denote $N = N_{r_E,r}$. Assume that $\lambda = t\sigma$ for some $t \in \mathbb{Q}$. The leading order term of $B_2(E, k\lambda)$ in k is

$$p_{r*}\frac{c_1(\mathcal{L}_\lambda)^{N+2}}{(N+2)!} \equiv_1 D_{\lambda,r_E}\left(\frac{h_2(\lambda)h_2(E)}{r_E(r_E+1)} + \frac{c_2(\lambda)c_2(E)}{r_E(r_E-1)}\right), \tag{73}$$

by Theorem 22. The term $B_1(E, k\lambda)$ can be computed easily using the splitting principle. In general, we have

$$c_1(E^\lambda) = \operatorname{rank} E^\lambda c_1(\lambda)c_1(E)/r_E. \tag{74}$$

It suffices to verify that the k-linear term of $B_2(E, k\lambda)$ satisfies the claimed identity.

For any line bundle L on the base B, the Hirzebruch-Riemann-Roch formula applied to the vector bundle $(E \otimes L)^{k\lambda}$ yields

$$\begin{aligned}\chi(B, E^{kt\sigma}) &= \int_B \operatorname{ch} L^{k|\lambda|} \operatorname{ch} E^{k\lambda} \mathrm{Td}_B \\ &= \int_B \sum_{i=0}^{b} \frac{(k|\lambda|c_1(L))^i}{i!} \operatorname{ch} E^{k\lambda} \mathrm{Td}_B.\end{aligned} \tag{75}$$

Moreover, we have

$$\frac{c_1(\mathcal{L}_\lambda(A))^{N+n}}{(N+n)!} = \sum_{i=1}^{n} \frac{c_1(\mathcal{L}_\lambda)^{N+i}}{(N+i)!} p^* \frac{c_1(A)^{n-i}}{(n-i)!} \tag{76}$$

for all $n \geq 1$ and $A \in \operatorname{Pic} B$.

By the asymptotic Hirzebruch-Riemann-Roch formula on $\mathcal{F}l_r(E)$ for the line bundle $\mathcal{L}_\lambda(L^{|\lambda|})^{\otimes k}$, we have

$$\begin{aligned}\chi(\mathcal{F}l_r(E), \mathcal{L}_\lambda(L^{|\lambda|})^k) = \int_{\mathcal{F}l_r(E)} \Bigg(&\frac{c_1(\mathcal{L}_\lambda(L^{|\lambda|}))^{N+b}}{(N+b)!} k^{N+b} \\ &- \frac{c_1(\mathcal{L}_\lambda(L^{|\lambda|}))^{N+b-1} K_{\mathcal{F}l_r(E)}}{2(N+b-1)!} k^{N+b-1}\Bigg) + O(k^{N+b-2})\end{aligned} \tag{77}$$

The remaining part of the statement then follows by comparing the k-degree $b-1$ coefficients of the $c_1(L)^{b-2}$ term in Equation (75) and Equation (77), latter of which is equal to

$$k^{N+1}\int_X \left(\frac{(p_{r*}c_1(\mathcal{L}_\lambda))^{N+2}}{2t(N+1)!} - \frac{(p_{r*}c_1(\mathcal{L}_\sigma))^{N+1}\left(c_1((\det E)^{\otimes l(\lambda)}) + K_B\right)}{2(N+1)!} \right) \frac{c_1(L^{b-2})}{(b-2)!}. \tag{78}$$

by Lemma 27. We write

$$B_2(E,k\lambda) = k^2 B_{2,2} + kB_{2,1} + O(k^0) \tag{79}$$

and expand the Chern character in of $E^{k\lambda}$ as

$$\operatorname{ch} E^{k\lambda} = D_{\lambda,r_E}\left(k^N + \frac{N}{2t}k^{N-1} + O(k^0)\right)\left(1 + B_1(E,k\lambda) + k^2B_{2,2} + kB_{2,1}\right). \tag{80}$$

We can see that

$$B_{2,1} = \left(\frac{h_2(\lambda)h_2(E)}{tr_E(r_E+1)} + \frac{c_2(\lambda)c_2(E)}{tr_E(r_E-1)} - \frac{l(\lambda)|\lambda|c_1(E)^2}{2r_E}\right), \tag{81}$$

which can be written as

$$\frac{t\left((r_E-1)h_2(\sigma) - (r_E+1)c_2(\sigma)\right)}{2r_E}\left(\frac{h_2(E)}{r_E(r_E+1)} - \frac{c_2(E)}{r_E(r_E+1)}\right), \tag{82}$$

Finally by Lemma 28 we have

$$\begin{aligned} \frac{(r_E-1)h_2(\sigma) - (r_E+1)c_2(\sigma)}{2r_E} &= h_2(\sigma) - \frac{(r_E+1)er_c^2}{2} \\ &= \frac{\sum_i r_i r_{i+1}(r_{i+1}-r_i)}{2} \\ &= \frac{t\left(e|\sigma| - \sum_i (2i-1)\sigma_i\right)}{r_E-1} \\ &= \frac{e|\lambda| - \sum_i(2i-1)\lambda_i}{r_E-1} \end{aligned} \tag{83}$$

This completes the proof. □

Remark 29. In general, there is a simple relation between the classes $B_{2,0}(\lambda,E)$ and $A_2(E)$. Namely we have

$$B_{2,0} - \frac{2(r_E+1)}{r_E-1}A_2(\lambda)A_2(E) = \frac{c_1(\lambda)^2c_1(E)^2}{2{r_E}^2}. \tag{84}$$

Remark 30. The same calculation can be used to find the codegree 1 asymptotics of $B_i(E,k\lambda)$ in any Chow degree, when $\lambda = k\sigma$ for some $k \in \mathbb{Q}$. Keeping to the same notation as in the proof, we have

$$\begin{aligned} B_m(E,k\lambda) = {} & k^m C_{\lambda,r}\frac{\sum_{|\mu|=m} s_\mu(\lambda)s_\mu(E)}{\prod_{i=1}^l (r_E+\mu_i-i)!} \\ & + k^{m-1}C_{\lambda,r}\left(\frac{m\sum_{|\mu|=m} s_\mu(\lambda)s_\mu(E)}{2t\prod_{i=1}^l(r_E+\mu_i-i)!} - \frac{|\lambda|c_1(E)\sum_{|\mu|=m-1} s_\pi(\lambda)s_\pi(E)}{2\prod_{i=1}^l(r_E+p_i-i)!}\right) \\ & + O(k^{m-2}), \end{aligned} \tag{85}$$

for any $m \geq 2$.

4 Flag variety over a curve

The aim of this section is to prove Theorem 1.

Proof of Theorem 1. Let B be a curve. Let F be a subbundle of E and A a line bundle on B such that the polarised scheme $(\mathscr{Y}, \mathscr{L}_\lambda(A))$, where $\mathscr{Y} = \mathcal{F}l_r(\mathcal{E}^{\mathcal{F}})$, is a test configuration for $(\mathcal{F}l_r(E), \mathcal{L}(A))$. We may assume that $\widetilde{E}^\lambda \otimes A$ is ample, since twisting by the pullback $\mathcal{O}_{\mathbb{P}^1}(1)$ leaves Equation (19) invariant. We will show that the Donaldson-Futaki invariant of the test configuration $(\mathscr{Y}, \mathscr{L}_\lambda(\pi^* A))$ satisfies

$$\mathrm{DF}(\mathscr{Y}) = C_{g,E,A,\lambda}(\mu_E - \mu_F), \tag{86}$$

where C is a positive number depending on B, A, E, F and λ. By Riemann-Roch the Hilbert polynomial of $\mathcal{L}_\lambda^k(A)$ satisfies

$$\chi(\mathcal{F}l_r(E), \mathcal{L}^k) = \operatorname{rank} E^{k\lambda} \left(a_0 k + a_1\right), \tag{87}$$

where

$$\begin{aligned} a_0 &= c_1(\lambda)\mu_E + \mu_A, \\ a_1 &= 1 - g. \end{aligned} \tag{88}$$

Using the Riemann-Roch formula on $B \times \mathbb{P}^1$, we can write

$$\chi(B, E^\lambda \otimes L^{mk}) = \int_{B\times\mathbb{P}^1} {r_E}^{kc_1(A)} \operatorname{ch}(\widetilde{E}^{k\lambda}) \mathrm{Td}_{B\times\mathbb{P}^1}. \tag{89}$$

By Theorem 16 we have

$$h^0(B \times \mathbb{P}^1, \widetilde{E}^{k\lambda}) = \operatorname{rank} E^{k\lambda}(b_0 k^2 + b_1 k + O(1)), \tag{90}$$

where denoted

$$b_0 = \frac{h_2(\lambda)h_2(\widetilde{E})}{r_E(r_E+1)} + \frac{c_2(\lambda)c_2(\widetilde{E})}{r_E(r_E-1)} + \frac{c_1(\lambda)}{r_E} c_1(\widetilde{E}).c_1(A) \tag{91}$$

and

$$b_1 = H_\lambda A_2(\widetilde{E}) - \frac{c_1(\lambda)c_1(\widetilde{E}).K_{B\times\mathbb{P}^1}}{2r_E} - \frac{c_1(A).K_{B\times\mathbb{P}^1}}{2}. \tag{92}$$

Here the class $A_2(\widetilde{E})$ is defined in Equation (35) and we write

$$H_\lambda = \frac{r_E c_1(\lambda) - \sum_{i=1}^{r_c}(2i-1)\lambda_i}{r_E - 1}. \tag{93}$$

Let $\mathbf{g}$ and $\mathbf{h}$ be the two fibres of the first and second projection of the product $B \times \mathbb{P}^1$, respectively. The intersection matrix with respect to this basis is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{94}$$

As a special case of Lemma 12 we have

$$c_1(\widetilde{E})^2 = 2r_F r_E \mu_E. \tag{95}$$

Calculating the intersection classes appearing in Equations (91) and (92) gives

$$\begin{aligned}
-\frac{c_1(\widetilde{E}).K_{B\times\mathbb{P}^1}}{2} &= (f\mathbf{h} + (r_E\mu_E)\mathbf{g}).(\mathbf{h} + (1-g)\mathbf{g}) \\
&= r_E\mu_E + \frac{(1-g)c_1(\widetilde{E})^2}{2r_E\mu_E}, \\
-\frac{c_1(A).K_{B\times\mathbb{P}^1}}{2} &= \mu_A\mathbf{g}.(\mathbf{h} + (1-g)\mathbf{g}) = \mu_A, \text{ and} \\
c_1(\widetilde{E}).c_1(A) &= r_F\mu_A.
\end{aligned} \tag{96}$$

Let $y = (y_1, \ldots, y_l)$ be variables. For any such y define the symmetric polynomial

$$A_2(y) = \frac{r_E - 1}{2}\left(\frac{h_2(y)}{r_E(r_E+1)} - \frac{c_2(y)}{r_E(r_E-1)}\right). \tag{97}$$

Using the above calculations and Remark 29 we then have

$$\begin{aligned}
b_0 &= \frac{2(r_E+1)}{r_E-1}A_2(\lambda)A_2(\widetilde{E}) + \frac{c_1(\lambda)^2 c_1(\widetilde{E})^2}{2r_E^2} + \frac{c_1(\lambda)r_F\mu_A}{r_E}, \\
b_1 &= H_\lambda A_2(\widetilde{E}) + a_0 + \frac{(1-g)c_1(\lambda)c_1^2(\widetilde{E})}{2r_E^2\mu_E},
\end{aligned} \tag{98}$$

By direct calculation, and Lemma 12 the Donaldson-Futaki invariant defined in Equation (19) is given by

$$\begin{aligned}
\mathrm{DF}(\mathscr{Y}) &= \left(a_1 b_0 - a_0 b_1 + a_0^2\right)/a_0^2 \\
&= C_{g,E,A,\lambda}(\mu_E - \mu_F),
\end{aligned} \tag{99}$$

where the constant $C_{g,E,A,\lambda}$ is given by

$$C_{g,E,A,\lambda} = \frac{r_F}{(r_E+1)\left(c_1(\lambda)\mu_E + \mu_A\right)^2}\left(H_\lambda\left(c_1(\lambda)\mu_E + \mu_A\right) + \frac{2(g-1)(r_E+1)A_2(\lambda)}{r_E-1}\right). \tag{100}$$

We are left to verify that the constant $C_{g,E,A,\lambda}$ is positive. For $g \geq 1$, it suffices to show that H_λ and $A_2(\lambda)$ are positive since $c_1(\lambda)\mu_E + \mu_A$ is positive as $\mathcal{L}_\lambda(A)$ is ample.

Using $r_E - 1 \geq r_c$ and recalling that r_c is the length of λ, we have

$$\begin{aligned}
(r_E+1)r_E(r_E-1)A_2(\lambda) &= (r_E-1)c_1(\lambda)^2 - 2r_E c_2(\lambda) \\
&= (r_E-1)\sum_{i=1}^{r_c}\lambda_i^2 - 2\sum_{1\leq i<j\leq r_c}\lambda_i\lambda_j \\
&\geq \sum_{1\leq i<j\leq r_c}(\lambda_i - \lambda_j)^2 > 0.
\end{aligned} \tag{101}$$

We have

$$\sum_{i=1}^{l}(2i-1)\lambda_i = \sum_{j=1}^{c}(\lambda_i')^2, \tag{102}$$

where λ' denotes the conjugate partition of λ. To see that the first term of Equation (100) is positive, notice that

$$ec_1(\lambda) - \sum_i (2i-1)\lambda_i = \sum_{j=1}^{s} \lambda_i'(r_E - \lambda_i') > 0, \tag{103}$$

which is positive since $r_E > r_c \geq \lambda_i'$ for all i. Hence $C_{g,E,A,\lambda} > 0$ for all $g \geq 1$. A similar calculation shows that $C_{0,E,A,\lambda}$ is positive. □

5 Flag variety over a base of higher dimension

Our aim is to prove Theorem 2. We proceed in two stages. First, we assume for simplicity that the test configuration is induced by a subsheaf of E. Finally, we use Proposition 32 that this can be done without loss of generality.

Proof of Theorem 2. By Proposition 32 we may assume that F is a subbundle. We will show that the leading term in m in the Donaldson-Futaki invariant of the test configuration $(\mathscr{Y}, \mathscr{L}_\lambda(p_1^* L^m))$ is

$$D_{E,\lambda,L,r_F}(\mu(E) - \mu(F)), \tag{104}$$

where D_{E,λ,L,r_F} is a positive number depending on B, L, E, F and λ. Here p_1 is the first projection from $B \times \mathbb{A}^1$. Expand the Chern character of $E^{k\lambda}$ as

$$\operatorname{ch} E^{k\lambda} = \sum_{i=0}^{b} \operatorname{ch}_i E^{k\lambda} \tag{105}$$

and the Todd class of B as

$$\operatorname{Todd}(B) = \sum_{i=0}^{b} \operatorname{Todd}_i(B). \tag{106}$$

We then have

$$\begin{aligned} \chi(\mathcal{F}l_r(E), \mathcal{L}_\lambda(L^m)^{\otimes k}) &= \chi(B, E^{k\lambda} \otimes L^{mk}) \\ &= \int_B {r_E}^{mk\omega} \operatorname{ch}(E^{k\lambda}) \operatorname{Td}(B) \\ &= \frac{(mk)^b}{b!}\omega^b \operatorname{rank}(E^{k\lambda}) \\ &+ \frac{(mk)^{b-1}}{(b-1)!}\omega^{b-1}\Big(\operatorname{rank}(E^{k\lambda})\frac{c_1(B)}{2} + \frac{kc_1(\lambda)c_1(E^\lambda)}{r_E}\Big) \\ &+ \frac{(mk)^{b-2}}{(b-2)!}\omega^{b-2}\Big(\operatorname{rank}(E^{k\lambda}) \operatorname{Todd}_2(B) + \frac{kc_1(\lambda)c_1(E^\lambda).c_1(B)}{2r_E} + \operatorname{ch}_2(E^{k\lambda})\Big) \\ &+ O(k^{b-3}), \end{aligned} \tag{107}$$

which follows from Riemann-Roch and the pushforward formula of Proposition **??**. Here $\mathrm{Td}_2(B)$ is the second Todd class of B. Using Riemann-Roch on $B \times \mathbb{P}^1$, we similarly compute the Hilbert polynomial of $\widetilde{E}^\lambda \otimes p_1^* L^m$, where p_1 is the first projection.

To apply Lemma 11, choose m_0 so that the bundle $E \otimes L^{\frac{m_0}{c_1(\lambda)}}$ is ample and assume from now on that $m > m_0$.

As in Section 4, we write

$$\begin{aligned} h^0(B, E^{k\lambda} \otimes L^{mk}) &= \operatorname{rank} E^{k\lambda} \left(a_0 k^b + a_1 k^{b-1} + O(k^{b-2})\right), \\ h^0(B \times \mathbb{P}^1, \widetilde{E}^{k\lambda} \otimes L^{mk}) &= \operatorname{rank} E^{k\lambda} \left(b_0 k^{b+1} + b_1 k^b + O(k^{b-1})\right). \end{aligned} \tag{108}$$

Next, we expand the a_i and the b_i in powers of m as

$$b_0 = b_{0,0} m^b + b_{0,1} m^{b-1} + O(m^{b-2}), \tag{109}$$

$$b_1 = b_{1,0} m^b + b_{1,1} m^{b-1} + O(m^{b-2}), \tag{110}$$

$$a_0 = a_{0,0} m^b + a_{0,1} m^{b-1} + O(m^{b-2}), \tag{111}$$

$$a_1 = a_{1,0} m^b + a_{1,1} m^{b-1} + O(m^{b-2}). \tag{112}$$

Let $\omega = c_1(L)$ and $\eta = p_1^* \omega$. Using Theorem 16 and equation (107), we see that

$$\begin{aligned}
b_{0,0} &= \frac{c_1(\lambda)}{r_E \cdot b!} \eta^b . c_1(\widetilde{E}) \\
b_{0,1} &= \frac{1}{(b-1)!} \eta^{b-1} . \left(\frac{h_2(\lambda) h_2(\widetilde{E})}{r_E(r_E+1)} + \frac{c_2(\lambda) c_2(\widetilde{E})}{r_E(r_E-1)} \right) \\
b_{1,0} &= -\frac{\eta^b . K_{B \times \mathbb{P}^1}}{2 \cdot b!} \\
b_{1,1} &= \frac{1}{(b-1)!} \left(\eta^{b-1} . H_\lambda A_2(\widetilde{E}) - \frac{c_1(\lambda) \eta^{b-1} . K_{B \times \mathbb{P}^1} . c_1(\widetilde{E})}{2 r_E} \right) \\
a_{0,0} &= \frac{\omega^b}{b!} = \frac{\deg L}{b!} \\
a_{0,1} &= \frac{c_1(\lambda)}{r_E (b-1)!} \omega^{b-1} . c_1(E) \\
a_{1,0} &= 0 \\
a_{1,1} &= -\frac{\omega^{b-1} . K_B}{2(b-1)!} = -\frac{\deg K_B}{2(b-1)!}.
\end{aligned}$$

The proof of the following lemma is a straightforward calculation.

Lemma 31. *The intersection numbers appearing above are*

$$\begin{aligned}
\omega^b &= \deg L \\
\omega^{b-1}.c_1(E) &= r_E\mu_E \\
\omega^{b-1}.K_B &= \deg K_B \\
\eta^b.c_1(\widetilde{E}) &= \deg L(r_E\alpha + r_E) \\
\eta^{b-1}.c_1(\widetilde{E})^2 &= 2r_F r_E \mu_E \\
\eta^{b-1}.c_2(\widetilde{E}) &= r_F r_E \mu_E - r_F\mu_F \\
\eta^{b-1}.K_{B\times\mathbb{P}^1}.c_1(\widetilde{E}) &= f\deg K_B - 2r_E\mu_E \\
\eta^b.K_{B\times\mathbb{P}^1} &= -2\deg L \\
\eta^b.A_2(\widetilde{E}) &= \frac{r_E(\mu_E-\mu_F)}{r_E+1}.
\end{aligned}$$

We write Laurent expansion of the Donaldson-Futaki invariant in m

$$\mathrm{DF}(\mathscr{Y},\mathcal{L}_{\mathcal{E},m},\rho) = F_0 + F_1 m^{-1} + O(m^{-2}), \tag{113}$$

where

$$F_0 a_0^2 = \underbrace{a_{1,0}b_{0,0}}_{=0} - a_{0,0}b_{1,0} + a_{0,0}^2 = -\left(\frac{\deg L}{b!}\right)^2 + \left(\frac{\deg L}{b!}\right)^2 = 0 \tag{114}$$

and

$$F_1 a_0^2 = \underbrace{a_{1,0}b_{0,1}}_{=0} + a_{1,1}b_{0,0} - a_{0,1}b_{1,0} - b_{1,1}a_{0,0} + 2a_{0,0}a_{0,1}. \tag{115}$$

An elementary calculation similar to the one we did in Section 4 shows that

$$\mathrm{DF}(\mathscr{Y},\mathscr{L}_\lambda(p_1^*L^m)) = D_{E,\lambda,L,r_E}(\mu_E-\mu_F)m^{-1} + O(m^{-2}) \tag{116}$$

where

$$D_{E,\lambda,L,r_E} = \frac{r_F b H_\lambda}{(r_E+1)\deg L} \tag{117}$$

is a positive constant by the same argument as in Section 4. Theorem 2 then follows from the following Proposition. □

Proposition 32. *Using notation from Section 2, let $(\mathcal{F}l_r(\mathcal{E}^{\mathcal{F}}),\mathscr{L}_\lambda(L^m))$ be a test configuration for $(\mathcal{F}l_r(E),\mathcal{L}_\lambda(L^m))$ where $\mathcal{F}$ is a saturated torsion free subsheaf of E. Then the formula*

$$\mathrm{DF}(\mathcal{F}l_r(\mathcal{E}^{\mathcal{F}}),\mathscr{L}_\lambda(L^m)) = D_{E,\lambda,L,r_E}(\mu_E-\mu_{\mathcal{F}})m^{-1} + O(m^{-2}) \tag{118}$$

for the Donaldson-Futaki invariant still holds for $m \gg 0$.

Proof. It follows that $E/\mathcal{F}$ is also torsion free, and $\mathcal{F}$ and $E/\mathcal{F}$ are both locally free over an open subset U whose complement is of dimension at least 2. The leading order terms in m of $h(k)$ and $w(k)$ given in Equation (109) only involve the first Chern classes of $\mathcal{F}$ and $E/\mathcal{F}$. But the first Chern classes can be computed over the open set U where F and $E/\mathcal{F}$ are locally free. The Schur functor commutes with localisation, so Theorem 16 holds for the restriction $\left(\mathcal{F} \oplus E/\mathcal{F}^{\lambda}\right)\big|_U$. Therefore, we may assume without loss of generality that $\mathcal{F}$ is a subbundle. □

6 K-stability of complete intersections

The objective of this section is to provide additional examples of K-unstable varieties. We describe a situation in which the Donaldson-Futaki invariant of a complete intersection can be calculated. In Section 7 and apply the result in the case of flag bundles in Section 8.

The idea is to fix a complete intersection X in a polarised variety Y and a test configuration $\mathscr{Y}$ for Y. Consider then the Zariski closure of the orbit of X in $\mathscr{Y}$ under the $\mathbb{G}_m$-action. The scheme $\mathscr{X}$ is a test configuration for X and its Donaldson-Futaki invariant depends, a priori, on the test configuration $\mathscr{Y}$ in a complicated way. However, in some favourable situations the Donaldson-Futaki invariant of $\mathscr{X}$ is related to the Donaldson-Futaki invariant of $\mathscr{Y}$ and topological data of X in Y. Examples of this behaviour have been given by Stoppa-Tenni [12] and Arezzo-Della Vedova [1].

The main result of this chapter is a generalisation of an example in [12].

Theorem 33 (A simple limit for high genus curves)**.** *Let E be an ample vector bundle of rank r_E on a curve, and F is a subbundle of E of rank r_F. Assume that*

$$(\mathscr{Y}, \mathscr{L}) = (\mathcal{F}l_r(\mathcal{E}^{\mathcal{F}}), \mathcal{L}_\lambda) \tag{119}$$

is a test configuration for $(\mathcal{F}l_r(E), \mathcal{L}_\lambda)$ as defined in Chapter 1, and that λ is in $\mathcal{P}_\diamond(r)$. Let X be a generic complete intersection in $\mathcal{F}l_r(E)$ of codimension less than the integer N_{λ, r_E, r_F} defined in Equation (127)*. Then the Donaldson-Futaki invariant of the test configuration $\mathscr{X}$, defined as the closure of the orbit of X in $\mathscr{Y}$, is given by*

$$\mathrm{DF}(\mathscr{X}) = D\left(C_E \deg E + C_F \deg F\right) g + O(g^0), \tag{120}$$

where D is a positive number and C_E and C_F are given in Equation (146)*. All three numbers depend only on* $\deg E, \deg F$*, the codimension u of X and λ.*

We may easily construct examples of K-unstable complete intersections in flag bundles over curves using Theorem 33. The simplest such construction is due to Stoppa and Tenni.

Fix a positive integer d and let $C(g)$ be a sequence of d-gonal curves of genus g for all integers g larger than 2, and let L_g be a degree d line bundle on $C(g)$. Let

$$F_g = L_g \text{ and } E_g = \mathcal{O}_{C(g)}^{\oplus r_E - 1} \oplus L_g.$$

With these choices $\deg E_g$ and $\deg F_g$ are bounded as functions of g and the final term in Equation (120) is under control. The vector bundle E_g is only globally generated but we may find a test configuration for an ample polarisation on X whose Donaldson-Futaki invariant is arbitrarily close to the one given by Equation (120) when applied to the globally generated vector bundle E_g. We do this by replacing the vector bundle E_g with $E_g \otimes A^{\frac{\epsilon}{|\lambda|}}$, where A is an ample line bundle on $C(g)$. Finally, we use the following Lemma which follows directly from calculations done in Sections 4 and 7.

Lemma 34. *The Donaldson-Futaki invariant of $(\mathscr{Y}_F, \mathscr{L}_\lambda(\epsilon A))$ is continuous in ϵ.*

Using Lemma 34 and simple combinatorics outlined in Section 8 we obtain the following new examples of K-unstable varieties.

Theorem 35 (Theorem **??**)**.** *Let Y be the Grassmannian of p-dimensional quotients of E_g with the polarisation $\mathcal{L}_\lambda(\epsilon A)$, where $\lambda = (1^p)$. Let s be a positive integer.*

Then there exists numbers $\epsilon_0 > 0$ and $g_0 > 0$ such that a general hypersurface H in Y which is a defined by a section of a multiple of $s\left(\mathcal{L}_\lambda(\epsilon A)\right)$ with the polarisation $\mathcal{L}_\lambda(\epsilon A)\big|_H$ is K-unstable for all $\epsilon < \epsilon_0$ and $g > g_0$.

We may also ask for H to be smooth in the statement of Theorem 35 by Bertini's theorem [6, Theorem II.8.18].

Proposition 36. *For $s > e$ the hypersurface H is of general type.*

Proof. We prove that K_H is ample. This follows directly from the adjunction formula [3, Example 3.2.12]. In the notation of Theorem 35, we have

$$K_H = \left(\mathcal{L}_{-\sigma} + K_{C(g)} + s\mathcal{L}_\lambda(\epsilon A)\right)\big|_H, \tag{121}$$

where σ is the partition $({r_E}^p)$. The statement then follows from Remark **??** and the preceding discussion. □

7 The Donaldson-Futaki invariant of a complete intersection

Let ρ be $\mathbb{G}_m$-action on a polarised variety (Y, L) of dimension n and let φ_i be sections of $H^0(Y, L^{s_i})$ for $1 \leq i \leq u$. Let γ be an integer, and assume that the natural representation of ρ on $H^0(Y, L^{s_i})$ acts on φ_i by $t.\varphi_i = t^{\gamma s_i}\varphi_i$ for all i and $t \in \mathbb{G}_m$. Denote the complete intersection of $\varphi_1, \ldots, \varphi_u$ by X. The $\mathbb{G}_m$-action determines a product test configurations $\mathscr{Y}$ for (Y, L) and $\mathscr{X}$ for $(X, L\big|_X)$, since X is invariant under ρ.

Write the Hilbert and weight functions of $\mathscr{Y}$ and $\mathscr{X}$ as

$$\begin{aligned} h^0_Y(k) &= a_0 k^n + a_1 k^{n-1} + O(k^{n-2}), \\ w_Y(k) &= b_0 k^{n+1} + b_1 k^n + O(k^{n-1}), \\ h^0_X(k) &= c_0 k^{n-u} + c_0 k^{n-u-1} + O(k^{n-u-2}) \end{aligned}$$

and

$$w_X(k) = d_0 k^{n-u+1} + c_0 k^{n-u} + O(k^{n-u-1}),$$

respectively. The following Proposition is a special case of [1, Theorem 4.1]. We present an elementary proof in Section **??** of the Appendix along the lines of [12].

Proposition 37. *The Donaldson-Futaki invariant of the test configuration $\mathscr{X}$ is given by*

$$\mathrm{DF}(\mathscr{X}) = \mathrm{DF}(\mathscr{Y}) + \frac{\nu_Y - \gamma}{n+1-u}\left(\frac{(n+1)S}{2u} - \frac{u\mu_Y}{n}\right), \tag{122}$$

where we have denoted

$$\nu_Y = \frac{b_0}{a_0}, \quad S = \sum_{i=1}^{u} s_i \quad \text{and} \quad \mu_Y = \frac{a_1}{a_0}.$$

The result of Proposition 37 also applies also to test configurations which are not products. Assume that $(\mathscr{Y}, \mathscr{L})$ is an arbitrary test configuration for (Y, L). Assume for simplicity that the exponent is 1. Let

$$R = \bigoplus_{k=0}^{\infty} R_k = \bigoplus_{k=0}^{\infty} H^0(Y, L^k) \tag{123}$$

be the graded coordinate ring of (Y, L) and let $F_\bullet R$ be a graded filtration corresponding to the test configuration $\mathscr{Y}$ (cf. Remark **??**). We have an induced map

$$R \longrightarrow R_\gamma := \bigoplus_{k=0}^{\infty} R_k / F_{n_k - 1} R_k, \tag{124}$$

where n_k is the smallest integer such that $F_{n_k} R_k = R_k$, which is finite by condition (iii) of Remark **??**. Let I_γ be the ideal generated by $\bigoplus_{k=0}^{\infty} F_{n_k-1} R_k$. Define the *subscheme of least weight* of the test configuration $\mathscr{Y}$ to be the subscheme of Y determined by R/I_γ.

The limit of the subscheme of least weight is fixed under the $\mathbb{G}_m$ action over the central fibre. Slightly more generally, the following lemma follows directly from the definition of the scheme Y_γ.

Lemma 38. *The closure of the orbit of the subscheme of least weight Y_γ in $\mathscr{Y}$ is isomorphic to $Y_\gamma \times \mathbb{A}^1$ as (quasi-projective) polarised varieties. Moreover, the lifting of the $\mathbb{G}_m$-action on $\mathbb{A}^1$ to $Y_\gamma \times \mathbb{A}^1$ is trivial with a possibly nontrivial linearisation.*

Proof. Let $\mathscr{Y}_\gamma$ denote the closure of Y_γ under the $\mathbb{G}_m$-action. Consider the linear map

$$\Phi \colon R \to \bigoplus_{k=0}^{\infty} \bigoplus_{i=0}^{\infty} \frac{F_i R_k / F_{i-1} R_k}{J} \tag{125}$$

defined by the projection $R_k \to R_k / F_{n_k - 1} R_k$ and J is generated by all the elements which lie in $\bigoplus_{k=0}^{\infty} F_{n_k-1} R_k$. It is straightforward to see that Φ is a homomorphism of graded rings whose kernel is exactly the ideal I_γ. Finally, the scheme $\mathscr{Y}_\gamma$ is isomorphic to the product $Y_\gamma \times \mathbb{A}^1$ since it is the projectivisation of the ring

$$\operatorname{Rees} F_\bullet R/\widetilde{J}, \tag{126}$$

where $\widetilde{J}$ is the ideal generated by $(\bigoplus_{i=1}^{n_k-1} F_i R)t^i$. The statement about the action follows since the $\mathbb{G}_m$-action simply scales any graded component of its coordinate ring with weight $-n_k$. □

Example 39. If the filtration $F_\bullet R$ is the slope filtration from Remark **??**, then the subscheme of least weight recovers the subscheme associated to the ideal $\mathscr{I} \subset \mathcal{O}_B$, in the notation of Remark **??**.

By a *generic* hypersurface or complete intersection, we mean one which is contained in a dense open set of the corresponding Hilbert scheme.

Lemma 40. *Let the dimension of the subscheme Y_γ be greater than or equal to u. Then a generic complete intersections of codimension u on Y degenerates to a complete intersection on the central fibre. Moreover, if φ is a generic section of $H^0(Y, L^s)$, then the limit of φ has weight $-n_s$ in the $\mathbb{G}_m$-representation on $H^0(\mathscr{Y}_0, \mathscr{L}_0)$.*

Proof. Let Z be a complete intersection in Y of codimension no larger than u. We can identify not just Y_γ, but $Z \cap Y_\gamma$, which is generically a proper intersection, with its limit in the central fibre of $\mathscr{Y}$. The locus $\mathcal{V}$ in the Hilbert scheme of complete intersections of the same topological type as Z, whose the intersection with Y_γ is not complete intersection, is determined by any finite set of generators of the ideal of Y_γ in Y. By the assumption on the codimension of Z, the locus $\mathcal{V}$ is a proper closed subset. Hence the locus where the limit is not a complete intersection is also a proper closed subset. The second claim follows from the definition of the $\mathbb{G}_m$-action. □

A nontrivial example where the above results can be applied is given in the following section.

8 Complete intersections in flag varieties

In this section we apply the results of Section 7 to flag bundles. Fix a smooth projective variety (B, L), a line bundle A on B and a flag bundle $Y = \mathcal{F}l_r(E)$ with an ample underlying vector bundle E of rank r_E. Let Y be polarised by its relative canonical bundle $\mathcal{L}_\sigma$. Fix a subsheaf $\mathcal{F} \subset E$ of rank r_F and let $(\mathscr{Y}, \mathscr{L}_\lambda(L))$ be the test configuration of $(Y, \mathcal{L}_\lambda(L))$ induced by the degeneration of the vector bundle E into a direct sum $\mathcal{F} \oplus E/\mathcal{F}$ defined in Section 2. We also denote $q = \operatorname{rank} E/\mathcal{F}$.

Lemma 41. *The relative dimension of the least weight subscheme in the central fibre $\mathcal{F}l_r(\mathcal{E}^{\mathcal{F}})_0$ is given by*

$$N_{r,r_E,r_F} = \sum_{i=1}^{p-2} r_i(r_{i+1} - r_i) + r_{p-1}(q - r_{p-1}) + \sum_{i=p}^{c} (r_i - q)(r_{i+1} - r_i), \tag{127}$$

where $r = (0, r_1, \ldots, r_c, r_E)$ and

$$p = \min\{a \colon e \geq a \geq 1, r_a > r_E - f\} \tag{128}$$

Proof. We will describe the filtration corresponding to the test configuration $\mathcal{F}l_r(\mathcal{E}^{\mathcal{F}})$ in detail in Section **??**. However, it suffices to see that the subscheme fixed by the $\mathbb{G}_m$-action on the central fibre is the intersection of $\mathcal{F}l_r(\mathcal{E}^{\mathcal{F}})$ with the subscheme

$$\prod_{i=1}^{p-1} \mathbb{P}(\bigwedge^{r_i} E/\mathcal{F}) \times \prod_{j=p}^{c} \mathbb{P}(\bigwedge^{q} E/\mathcal{F} \otimes \bigwedge^{r_j-q} E) \subset \prod_{k=1}^{c} \mathbb{P}(\bigwedge^{r_k} E) \tag{129}$$

The dimension of the locus of k-planes containing a fixed q-dimensional vector space in a Grassmannian of k-planes in an l-dimensional vector space is $(k-q)(l-k)$. The dimension in Equation (127) is then calculated by considering the flag bundle as an iterated fibration of Grassmannians and using elementary geometric considerations. □

Lemma 42. *Let λ be an element of $\mathcal{P}(r)$. The lowest weight γ of the $\mathbb{G}_m$-action on sections of $\mathcal{L}_\lambda$ is given by*

$$\gamma = \sum_{i=p}^{c} s_i \max\{(r_i - q), 0\}, \tag{130}$$

where $s_{c-i} = \lambda_i - \lambda_{i-1}$ for $i \in r$ and p was defined in Equation (128).

Proof. Recall that the bundle $\mathcal{L}_\lambda$ is the restriction of the line bundle $\bigotimes_{i=1}^{c} \mathcal{O}_{\mathbb{P}(\bigwedge^{r_i} E)}(s_i)$. By Borel-Weil (cf. Equation **??**) the sections of lowest weight over the central fibre of $\mathscr{Y}$ are sections of

$$\bigotimes_{i=1}^{p-1} S^{s_i}(\bigwedge^{r_i} E/\mathcal{F}) \otimes \bigotimes_{j=p}^{c} S^{s_j}\left(\bigwedge^{q} E/\mathcal{F} \otimes \bigwedge^{r_j-q} \mathcal{F}\right). \tag{131}$$

The statement of the Lemma follows by the definition of the action, which scales fibres of $\mathcal{F}$ by weight 1 and fixes the complement $E/\mathcal{F}$. □

For any tuple of sections

$$\underline{\varphi} = (\varphi_1, \ldots, \varphi_u) \in \prod_{i=1}^{q} |s_i \mathcal{L}_\lambda(A)| \tag{132}$$

we write

$$X_{\underline{\varphi}} = Z(\varphi_1) \cap \ldots \cap Z(\varphi_u) \tag{133}$$

for their intersection. Let $\mathscr{X}$ be the Zariski closure of the orbit of X under the $\mathbb{G}_m$-action inside $\mathscr{Y}$. Let $\mathcal{F}$ be a torsion free, saturated coherent subsheaf of E and assume that the sections φ_i are generic and that $u < N_{r,r_E,r_F}$. We are now in the situation of Lemma 40 and hence of Proposition 37 with the weight γ given by Lemma 42. We take the polarisation on $X_{\underline{\varphi}}$ to be the restriction $\mathcal{L}_\lambda(A)$.

We now revert to the notation of Sections 4 and 5, where b_0, b_1, a_0 and a_1 are the coefficients of the two highest degree terms of polynomials $\chi(\mathcal{F}l_r(E), \mathcal{L}_\lambda(A)^k)/\operatorname{rank} E^{k\lambda}$ and $\chi(B, \widetilde{E}^{k\lambda} \otimes A^k)/\operatorname{rank} E^{k\lambda}$, respectively. Recall that sections of $E^{k\lambda}$ correspond to sections of $\mathcal{L}_\lambda(A)^k$ and the highest order terms of the polynomial $\chi(B, \widetilde{E}^{k\lambda} \otimes A^k)$ and the weight polynomial $w(k)$ of $(\mathscr{Y}, \mathscr{L}_\lambda(A))$ agree.

Proposition 43. *Let σ be the canonical partition $\sigma_{r_F,r}$ (cf. Definition 14). The difference*

$$\Delta = \mathrm{DF}(\mathscr{Y}) - \mathrm{DF}(\mathscr{X}) \tag{134}$$

is positive for the polarisation $\mathcal{L}_\sigma$ if the base B is a curve. If the dimension $\dim_{\mathbb{C}} B$ is arbitrary, then Δ is positive when the polarisation is taken to be $\mathcal{L}_\sigma(L^m)$ on $\mathcal{F}l_r(E)$ for $m \gg 0$.

Remark 44. If B is a curve, E is ample and semistable, then the complete intersection $X_{\underline{\varphi}}$ polarised by the restriction of the bundle $\mathcal{L}_\sigma$ is not destabilised by test configurations induced from extensions of E.

If B is an arbitrary polarised manifold, the same statement is true for complete intersections of sections of $\mathcal{L}_\sigma(L^m)^{\otimes s_i}$, $1 < i < u$, for $m \gg 0$. It would be more interesting, although much harder, to study the asymptotics of test configurations of a fixed complete intersection as m goes to infinity.

Proof of Proposition 43. Indeed we have

$$\frac{b_0}{a_0} - \gamma \geq 0 \tag{135}$$

with equality only in the case of the action scaling every section with the same weight. The above inequality is equivalent to

$$\lim_{k\to\infty} \frac{w_Y(k)}{kh^0_Y(k)} - \gamma \geq 0 \tag{136}$$

where $h^0_Y(k)$ is Hilbert polynomial of $\mathcal{L}_\sigma(A)$ and $w(k)$ its equivariant analogue. Write

$$w(k) = \sum_i i \dim V_i^{(k)}, \tag{137}$$

where $V_i^{(k)}$ is the ith weight subspace of the representation of $\mathbb{G}_m$ on $H^0(B, E^{k\sigma} \otimes A^k)$. By definition of γ, we have

$$w_Y(k) \geq \sum_i \gamma \dim V_i^{(k)} = \gamma k h^0_Y(k). \tag{138}$$

It suffices to show that we have the inequality

$$\frac{n(n+1)}{2} \geq \mu_Y \tag{139}$$

We have

$$\mu_Y = \mu_{\mathbf{f}} + \mu_{\mathrm{rel}}, \tag{140}$$

where $\mu_{\mathbf{f}}$ is the *slope* of a fibre defined by

$$\mathrm{rank}\, E^{k\sigma} = D_{\sigma,r}\left(k^N_{r_E,r} + \mu_{\mathbf{f}} k^{N_{r_E,r}-1} + O(k^{N_{r_E,r}-2})\right), \tag{141}$$

for some rational number $D_{\sigma,r}$ and $\mu_{\mathrm{rel}} = \frac{a_1}{a_0}$. By the choice of polarisation we have $\mu_{\mathbf{f}} = \frac{N_{r_E,r}}{2}$. The other term μ_{rel} is obtained from Riemann-Roch. In the case $\dim B = 1$ the inequality (139) is clear. Consider the line bundle $\mathcal{L}_\sigma(L^m)$. Then by Equation (109) we have

$$\mu_{\mathrm{rel}} = -\frac{b \deg K_B}{2 \deg L} m^{-1} + O(m^{-2}), \tag{142}$$

so there is an $m_0 > 0$ such that the inequality (139) holds for $m > m_0$. □

In light of Proposition 43, we suspect that one has to start with an unstable vector bundle E in order to find K-unstable examples of complete intersections for some choices of the parameters E, F, B and s_i. We conclude with the proof of Theorem 33 and explain how Theorem 35 follows from Theorem 33.

Proof of Theorem 33. By Lemma 40, Lemma 41 and Lemma 42 we are in situation of Proposition 37, so the rest of the proof reduces to a straightforward calculation. Recall from Chapter 1 that we have

$$\begin{aligned}
b_0 &= \frac{h_2(\lambda) r_F (r_E \mu_E + \mu_F)}{r_E(r_E+1)} + \frac{c_2(\lambda) r_F (r_E \mu_E - \mu_F)}{r_E(r_E-1)}, \\
b_1 &= H_\lambda A_2(\widetilde{E}) + c_1(\lambda)\left(\mu_E + \frac{r_F}{r_E}(1-g)\right), \\
a_0 &= c_1(\lambda)\mu_E,
\end{aligned}$$

and

$$a_1 = 1 - g,$$

where $c_i(\lambda)$ denotes the ith elementary symmetric polynomial of $\lambda = (\lambda_1, \ldots, \lambda_c)$. After some algebraic manipulation we can write $\mathrm{DF}(\mathscr{X}) = Cg + O(g^0)$ where

$$C = D\Big(h_2(\lambda)(n+1)(n-u) r_F {r_E}^2 (\mu_E - \mu_F) - \gamma u c_1(\lambda)\mu_E \tag{143}$$

$$- c_1(\lambda)^2 r_E (r_E+1) r_F \left((n^2 + n - nu - r_E u)\mu_E - (n+1)(n-u)\mu_F)\right)\Big) \tag{144}$$

where $n = N_{r_E,r} + 1$ and $D = \left({r_E}^2({r_E}^2 - 1) n(n+1-u)\right)^{-1}$. Alternatively we can write

$$\mathrm{DF}(\mathscr{X}) = D\left(C_E \deg E + C_F \deg F\right) g + O(g^0), \tag{145}$$

where we have denoted

$$C_E = ({r_E}^2 - 1) u c_1(\lambda)(r_F c_1(\lambda) - r_E \gamma) - \frac{r_F}{r_E} C_F \tag{146}$$

and

$$C_F = (N_{r_E,r} + 1)(N_{r_E,r} - u)\left((r_E+1)c_1(\lambda)^2 - 2 r_E h_2(\lambda)\right). \tag{147}$$

□

Proof of Theorem 35. In the situation of Theorem 35 we have $\deg E = \deg F$. Computing the sign of the sum $C_E + C_F$ amounts to solving a polynomial inequality in e, λ, f and u. Let p be an integer between 1 and $e-1$. Since we are assuming $e - f \geq p$, we also have $\gamma = 0$ by Lemma 42. Then there exist positive constants D' and D'' such that

$$\begin{aligned} D'(C_E + C_F) = D''(u-1) \\ &- (r_E - r_F)(r_E - p - 1)(r_E - p)(r_E - p + 1)(p-1)p(p+1) \\ &- r_E(r_E - 1)(r_E + 1)(r_E - r_F - p)p. \end{aligned} \tag{148}$$

Hence assuming $u = 1$ implies immediately that $C_E + C_F < 0$ so the test configuration induced from $(\mathscr{Y}, \mathscr{L})$ as described on page 20. The code for repeating the calculations and for simulating more examples is contained in [7, Futaki invariants of complete intersections]. □

Remark 45. While the inequality $C_E + C_F < 0$ seems to hold more generally we only know how to prove it in the Grassmannian case.

Example 46 (Projective bundles)**.** Equation (143) gets a very nice form for projective bundles. In the notation used in the proof of Theorem 33, letting $\lambda = (1)$ gives

$$\mathrm{DF}(\mathscr{X}) = \Big(\frac{(r_F - \gamma u)\deg E - (r_E - u)\deg F}{{r_E}^2(r_E + 1 - u)}\Big) g + O(g^0). \tag{149}$$

This is the example given by Stoppa-Tenni [12]. Note that the convention the authors use for $\mathbb{P}E$ is dual to ours.

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